SPHERICAL

Cambridge:
PRINTED BY C. J. CLAY, M.A.
AT THE UNIVERSITY PRESS.

SPHERICAL T

For the Use of C

WITH NUMERO

]

I. TODHU

FELLOW AND ASSISTANT TUTOR O

Cam

MACMILL

AND 23, HENRIETTA STREE

1

PREFACE.

The present work is constructed on the same plan as my treatise on Plane Trigonometry, to which it is intended as a sequel; it contains all the propositions usually included under the head of Spherical Trigonometry, together with a large collection of examples for exercise. In the course of the work reference is made to preceding writers from whom assistance has been obtained; besides these writers I have consulted the treatises on Trigonometry by Lardner, Lefebure de Fourcy, and Snowball, and the treatise on Geometry published in the Library of Useful Knowledge. The examples have been chiefly selected from the University and College Examination Papers.

In the account of Napier's Rules of Circular Parts an explanation has been given of a method of proof devised by Napier, which seems to have been overlooked by most modern writers on the subject. I have had the advantage of access to an unprinted Memoir on this point by the late R. L. Ellis of Trinity College; Mr Ellis had in fact rediscovered for himself Napier's own method. For the use of this Memoir and for some valuable references on the subject I am indebted to the Dean of Ely.

Considerable labour has been bestowed on the text in order to render it comprehensive and accurate, and the examples have all been carefully verified; and thus I venture to hope that the work will be found useful by Students and Teachers.

I. TODHUNTER.

St. John's College,
August 15, 1859.

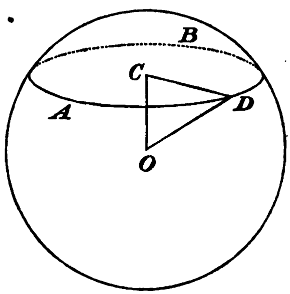
CONTENTS.

SPHERICAL TRIGONOMETRY.

I. GREAT AND SMALL CIRCLES.

1. A SPHERE is a solid bounded by a surface every point of which is equally distant from a fixed point which is called the *centre* of the sphere. The straight line which joins any point of the surface with the centre is called a *radius*. A straight line drawn through the centre and terminated both ways by the surface is called a *diameter*.

2. *The section of the surface of a sphere made by any plane is a circle.*



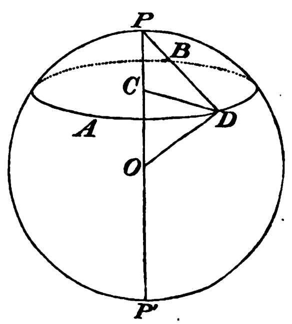
Let AB be the section of the surface of a sphere made by any plane, O the centre of the sphere. Draw OC perpendicular to the plane; take any point D in the section and join OD, CD. Since OC is perpendicular to the plane, the angle OCD is a right angle; therefore $CD = \sqrt{(OD^2 - OC^2)}$. Now O and C are fixed points, so that OC is constant; and OD is constant, being the radius of the sphere; hence CD is constant. Thus all points in the plane section are equally distant from the fixed point C; therefore the section is a circle of which C is the centre.

3. The section of the surface of a sphere by a plane is called a *great circle* if the plane passes through the centre of the sphere, and a *small circle* if the plane does not pass through the centre of the sphere. Thus the radius of a great circle is equal to the radius of the sphere.

4. Through the centre of a sphere and any two points on the surface a plane can be drawn; and only one plane can be drawn, except when the two points are the extremities of a diameter of the sphere, and then an infinite number of such planes can be drawn. Hence only one great circle can be drawn through two given points on the surface of a sphere, except when the points are the extremities of a diameter of the sphere. When only one great circle can be drawn through two given points, the great circle is unequally divided at the two points; we shall for brevity speak of the shorter of the two arcs as *the* arc of a great circle joining the two points.

5. The *axis* of any circle of a sphere is that diameter of the sphere which is perpendicular to the plane of the circle; the extremities of the axis are called the *poles* of the circle. The poles of a great circle are equally distant from the plane of the circle. The poles of a small circle are not equally distant from the plane of the circle; they may be called respectively the *nearer* and *further* pole; sometimes the nearer pole is for brevity called *the* pole.

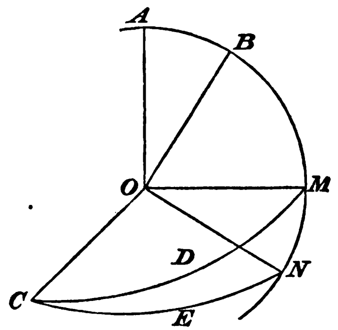
6. *A pole of a circle is equally distant from every point of the circumference of the circle.*



Let O be the centre of the sphere, AB any circle of the sphere, C the centre of the circle, P and P' the poles of the circle. Take any point D in the circumference of the circle; join CD, OD, PD. Then $PD = \sqrt{(PC^2 + CD^2)}$; and PC and CD are constant, therefore PD is constant. Suppose a great circle to pass through the points P and D; then the chord PD is constant, and therefore the arc of a great circle intercepted between P and D is constant for all positions of D on the circle AB.

Thus the distance of a pole of a circle from every point of the circumference of the circle is constant, whether that distance be measured by the straight line joining the points, or by the arc of a great circle intercepted between the points.

7. *The angle subtended at the centre of a sphere by the arc of a great circle which joins the poles of two great circles is equal to the inclination of the planes of the great circles.*



Let O be the centre of the sphere, CD, CE the great circles intersecting in C, A and B the poles of CD and CE respectively.

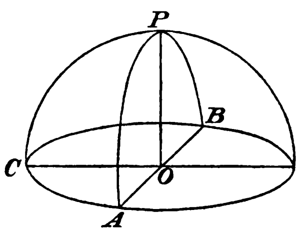
Draw a great circle through A and B, meeting CD and CE in M and N respectively. Then AO is perpendicular to OC, which is a line in the plane OCD; and BO is perpendicular to OC, which is a line in the plane OCE; therefore OC is perpendicular to the plane AOB (Euclid, XI. 4); and therefore OC is perpendicular to the lines OM and ON, which are in the plane AOB. Hence MON is the angle of inclination of the planes OCD and OCE. And the angle $AOB = AOM - BOM = BON - BOM = MON$.

8. By the angle between two great circles is meant *the angle of inclination of the planes of the circles.* Thus, in the figure of the preceding Article, the angle between the great circles CD and CE is the angle MON. In the figure to Art. 6, since PO is perpendicular to the plane ACB, every plane which contains PO is at right angles to the plane ACB. Hence the angle between any circle and a great circle which passes through its poles is a right angle.

9. *Two great circles bisect each other.*

For since the plane of each great circle passes through the centre of the sphere, the line of intersection of these planes is a diameter of the sphere, and therefore also a diameter of each great circle; therefore the great circles are bisected at the points where they meet.

10. *The arc of a great circle which is drawn from a pole of a great circle to any point in its circumference is a quadrant, and is at right angles to the circumference.*



Let P be a pole of the great circle ABC; the arc PA is a quadrant and at right angles to ABC.

For let O be the centre of the sphere, and draw PO. Then PO is at right angles to the plane ABC, because P is the pole of

ABC, therefore *POA* is a right angle, and the arc *PA* is a quadrant. And because *PO* is at right angles to the plane *ABC*, the angle between the planes *POA* and *ABC* is a right angle; therefore the arc *PA* is at right angles to *AC*.

11. *If the arcs of great circles joining a point* P *on the surface of a sphere with two other points* A *and* C *on the surface of the sphere which are not at opposite extremities of a diameter be each of them equal to a quadrant,* P *is a pole of the great circle through* A *and* C. (See figure of preceding Article.)

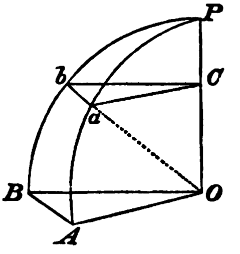
For suppose *PA* and *PC* to be quadrants and *O* the centre of the sphere; then since *PA* and *PC* are quadrants, the angles *POC* and *POA* are right angles. Hence *PO* is at right angles to the plane *AOC*, and *P* is a pole of the great circle *AC*.

12. Great circles which pass through the poles of a great circle are called *secondaries* to that circle. Thus, in the figure of Art. 7 the point *C* is a pole of *ABMN*, and therefore *CM* and *CN* are parts of secondaries to *ABMN*. And the angle between *CM* and *CN* is measured by *MN*; that is, *the angle between any two great circles is measured by the arc they intercept on the great circle to which they are secondaries.*

13. *If from a point on the surface of a sphere there can be drawn two arcs of great circles, not parts of the same great circle, which are at right angles to a given circle, that point is a pole of the circle.*

For, since the two arcs are at right angles to the given circle, the planes of these arcs are at right angles to the plane of the given circle, and therefore the line in which they intersect is perpendicular to the plane of the given circle, and is therefore the axis of the given circle; hence the point from which the arcs are drawn is a pole of the circle.

14. *To compare the arc of a small circle subtending any angle at the centre of the circle with the arc of a great circle subtending the same angle at its centre.*



Let ab be the arc of a small circle, C the centre of the circle, P the pole of the circle, O the centre of the sphere. Through P draw the great circles PaA and PbB, meeting the great circle of which P is a pole, in A and B respectively; draw Ca, Cb, OA, OB. Then Ca, Cb, OA, OB are all perpendicular to OP, because the planes aCb and AOB are perpendicular to OP; therefore Ca is parallel to OA, and Cb is parallel to OB. Therefore the angle aCb = the angle AOB (Euclid, XI. 10). Hence,

$$\frac{\text{arc } ab}{\text{radius } Ca} = \frac{\text{arc } AB}{\text{radius } OA}, \text{ (\textit{Plane Trigonometry}, Art. 18)};$$

therefore,
$$\frac{\text{arc } ab}{\text{arc } AB} = \frac{Ca}{OA} = \frac{Ca}{Oa} = \sin POa.$$

II. SPHERICAL TRIANGLES.

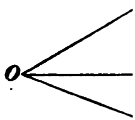
15. Spherical Trigonometry investigates the relations which subsist between the angles of the plane faces which form a solid angle and the angles at which the plane faces are inclined to each other.

16. Suppose that the angular point of a solid angle is made the centre of a sphere; then the planes which form the solid angle

will cut the sphere in arcs of
formed on the surface of the
triangle if it is bounded by *th*
the case when the solid angle
plane angles. If the solid ar
more than three plane angles,
surface of the sphere is bounde
circles, and is called a *spherical*

17. The three arcs of gr
triangle are called the *sides* of
formed by the arcs at the poi
angles of the spherical triangle.

18. Thus, let O be the cen
angle formed at O by the m



AB, BC, CA be the arcs of gr
the sphere; then ABC is a sp
BC, CA are its sides. Suppos
AB, and Ac the tangent at A
drawn from A *towards* B and
is one of the angles of the sp
formed in like manner at B
spherical triangle.

19. The principal part of
consists of theorems relating to

necessary to obtain an accurate conception of a spherical triangle and its parts.

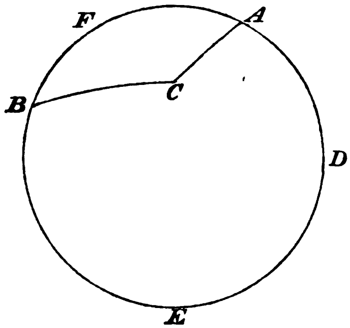
It will be seen that what are called *sides* of a spherical triangle are really *arcs* of great circles, and these arcs are proportional to the three plane angles which form the solid angle corresponding to the spherical triangle. Thus, in the figure of the preceding Article, the arc AB forms one side of the spherical triangle ABC, and the plane angle AOB is measured by the fraction $\frac{\text{arc } AB}{\text{radius } OA}$; and thus the arc AB is proportional to the angle AOB so long as we keep to the same sphere.

The *angles* of a spherical triangle are the inclinations of the plane faces which form the solid angle; for since Ab and Ac are both perpendicular to OA, the angle bAc is the angle of inclination of the planes OAB and OAC.

20. The letters A, B, C are generally used to denote the *angles* of a spherical triangle, and the letters a, b, c are used to denote the *sides*. As in the case of plane triangles, A, B, and C may be used to denote the numerical values of the angles expressed in *terms of any unit*, provided we understand distinctly what the unit is. Thus, if the angle C be a right angle, we may say that $C = 90°$, or that $C = \frac{\pi}{2}$, according as we adopt for the unit a degree or the angle subtended at the centre by an arc equal to the radius. So also, as the sides of a spherical triangle are proportional to the angles subtended at the centre of the sphere, we may use a, b, c to denote the numerical values of those angles in terms of any unit. We shall usually suppose both the angles and sides of a spherical triangle expressed in *circular measure*. (*Plane Trigonometry*, Art. 20.)

21. In future, unless the contrary be distinctly stated, any arc drawn on the surface of a sphere will be supposed to be an arc of a great circle.

22. In spherical triangles each side is restricted to be less than a semicircle; this is of course a *convention*, and it is adopted because it is found convenient.



Thus, in the figure, the arc $ADEB$ is greater than a semicircle, and we might, if we pleased, consider $ADEB$, AC and BC as forming a triangle, having its angular points at A, B, and C. But we agree to exclude such triangles from our consideration; and the triangle having its angular points at A, B, and C, will be understood to be that formed by AFB, BC and CA.

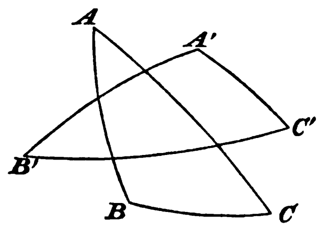
23. From the restriction of the preceding Article it will follow that *any angle of a spherical triangle is less than two right angles.*

For suppose a triangle formed by BC, CA and $BEDA$ having the angle BCA greater than two right angles. Then suppose D to denote the point in which the arc BC, if produced, will meet AE; then BED is a semicircle by Art. 9, and therefore BEA is greater than a semicircle; thus the proposed triangle is not one of those which we consider.

III. SPHERICAL GEOMETRY.

24. The relations between the sides and angles of a Spherical Triangle, which are investigated in treatises on Spherical Trigonometry, are chiefly such as involve the *Trigonometrical Functions* of the sides and angles. Before proceeding to these, however, we shall collect, under the head of Spherical Geometry, some theorems which involve the sides and angles *themselves*, and not their trigonometrical ratios.

25. *Polar Triangle.* Let ABC be any spherical triangle, and let the points A', B', C' be those poles of the arcs BC, CA, AB



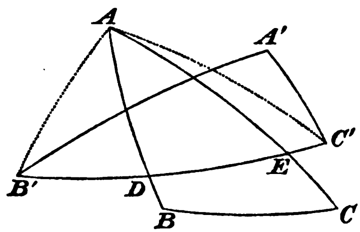
respectively which lie upon the same sides of them as the opposite angles A, B, C; then the triangle $A'B'C'$ is said to be the *polar triangle* of the triangle ABC.

Since there are two poles for each side of a spherical triangle, *eight* triangles can be formed having for their angular points poles of the sides of the given triangle; but there is only one triangle in which these poles A', B', C' lie towards the same parts with the corresponding angles A, B, C; and this is the triangle which is known under the name of the *polar triangle.*

The triangle ABC is called the *primitive* triangle with respect to the triangle $A'B'C'$.

26. *If one triangle be the polar triangle of another, the latter will be the polar triangle of the former.*

Let ABC be any triangle, $A'B'C'$ the polar triangle; then ABC will be the polar triangle of $A'B'C'$.



For since B' is a pole of AC, the arc AB' is a quadrant (Art. 10); and since C' is a pole of BA, the arc AC' is a quadrant; therefore A is a pole of $B'C'$ (Art. 11). Also A and A' are on the same side of $B'C'$; for A and A' are by hypothesis on the same side of BC, therefore $A'A$ is less than a quadrant; and since A is a pole of $B'C'$ and AA' is less than a quadrant, A and A' are on the same side of $B'C'$.

Similarly it may be shewn that B is a pole of $C'A'$, and that B and B' are on the same side of $C'A'$; also that C is a pole of $A'B'$, and that C and C' are on the same side of $A'B'$. Thus ABC is the polar triangle of $A'B'C'$.

27. *The sides and angles of the polar triangle are respectively the supplements of the angles and sides of the primitive triangle.*

For let the arc $B'C'$, produced if necessary, meet the arcs AB, AC, produced if necessary, in the points D and E respectively; then since A is a pole of $B'C'$, the spherical angle A is measured by the arc DE (Art. 12). But $B'E$ and $C'D$ are each quadrants; therefore DE and $B'C'$ are together equal to a semicircle; that is, the angle subtended by $B'C'$ at the centre of the sphere is the supplement of the angle A. This we may express for shortness thus; $B'C'$ is the supplement of A. Similarly it may be shewn that $C'A'$ is the supplement of B, and $A'B'$ the supplement of C.

And since ABC is the polar triangle of $A'B'C'$, it follows that BC, CA, AB are respectively the supplements of A', B', C'; that is, A', B', C' are respectively the supplements of BC, CA, AB.

From these properties a primitive triangle and its polar triangle are sometimes called *supplemental triangles.*

Thus, if A, B, C, a, b, c denote respectively the angles and sides of a spherical triangle all expressed in circular measure, and A', B', C', a', b', c' those of the polar triangle, we have

$$A' = \pi - a, \quad B' = \pi - b, \quad C' = \pi - c,$$

$$a' = \pi - A, \quad b' = \pi - B, \quad c' = \pi - C.$$

28. The preceding result is of great importance; for if any general theorem be demonstrated with respect to the sides and angles of any spherical triangle it holds of course for the polar triangle also. *Thus any such theorem will remain true when the angles are changed into the supplements of the corresponding sides and the sides into the supplements of the corresponding angles.* We shall see several examples of this principle in the next chapter.

29. *Any two sides of a spherical triangle are together greater than the third side.* (See fig. to Art. 18.)

For any two of the three plane angles which form the solid angle at O are together greater than the third (Euclid, XI. 20). Therefore any two of the arcs AB, BC, CA, are together greater than the third.

From this proposition it is obvious that any side of a spherical triangle is greater than the difference of the other two.

30. *The sum of the three sides of a spherical triangle is less than the circumference of a great circle.* (See fig. to Art. 18.)

For the sum of the three plane angles which form the solid angle at O is less than four right angles (Euclid, XI. 21); therefore

$$\frac{AB}{OA} + \frac{BC}{OA} + \frac{CA}{OA} \text{ is less than } 2\pi,$$

therefore, $AB + BC + CA$ is less than $2\pi \times OA$;

that is, the sum of the arcs is less than the circumference of a great circle.

31. The propositions contained in the preceding two Articles may be extended. Thus, if there be any polygon which has each of its angles less than two right angles, *any one side is less than the sum of all the others.* This may be proved by repeated use of Art. 29. Suppose, for example, the figure has four sides, and let the angular points be denoted by A, B, C, D. Then

$$AB + BC \text{ is greater than } AC;$$

therefore, $AB + BC + CD$ is greater than $AC + CD$,

and *à fortiori* greater than AD.

Again, if there be any polygon which has each of its angles less than two right angles, *the sum of its sides will be less than the circumference of a great circle.* This follows from Euclid, XI. 21, in the manner shewn in Art. 30.

32. *The three angles of a spherical triangle are together greater than two right angles and less than six right angles.*

Let A, B, C be the *angles* of a spherical triangle; let a', b', c' be the *sides* of the polar triangle. Then by Art. 30,

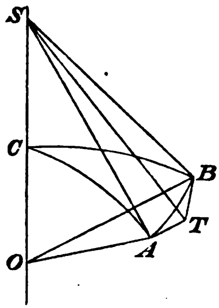
$$a' + b' + c' \text{ is less than } 2\pi,$$

that is, $\pi - A + \pi - B + \pi - C$ is less than 2π;

therefore, $A + B + C$ is greater than π.

And since each of the angles A, B, C is less than π, the sum $A + B + C$ is less than 3π.

33. *The angles at the base of an isosceles spherical triangle are equal.*



Let ABC be a spherical triangle having $AC = BC$; let O be the centre of the sphere. Draw tangents at the points A and B to the arcs AC and BC respectively; these will meet OC produced in the same point S, and AS will be equal to BS.

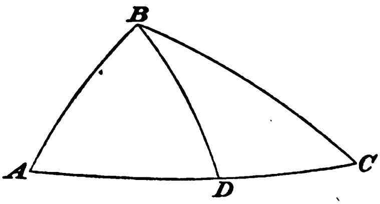
Draw tangents AT, BT at the points A, B to the arc AB; then $AT = TB$; join TS. In the two triangles SAT, SBT the sides SA, AT, TS are equal to SB, BT, TS respectively; therefore the angle SAT is equal to the angle SBT; and these are the angles at the base of the spherical triangle.

The figure supposes AC and BC to be less than quadrants; if they are greater than quadrants the tangents to AC and BC will meet on CO produced through O instead of through C, and the demonstration may be completed as before. If AC and BC are quadrants, the angles at the base are right angles by Arts. 8 and 11.

34. *If two angles of a spherical triangle are equal, the opposite sides are equal.*

Since the primitive triangle has two equal angles, the polar triangle has two equal sides; therefore in the polar triangle the angles opposite the equal sides are equal by Art. 33. Hence in the primitive triangle the sides opposite the equal angles are equal.

35. *If one angle of a spherical triangle be greater than another, the side opposite the greater angle is greater than the side opposite the other.*



Let ABC be a spherical triangle, and let the angle ABC be greater than the angle BAC; then the side AC will be greater than the side BC. At B make the angle ABD equal to the angle BAD; then BD is equal to AD (Art. 34), and $BD + DC$ is greater than BC (Art. 29); therefore $AD + DC$ is greater than BC; that is, AC is greater than BC.

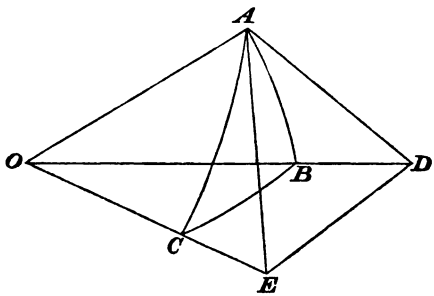
36. *If one side of a spherical triangle be greater than another, the angle opposite the greater side is greater than the angle opposite the other.*

This follows from the preceding Article by means of the polar triangle.

Or thus; suppose the side AC greater than the side BC, then the angle ABC will be greater than the angle BAC. For the angle ABC cannot be less than the angle BAC by Art. 35, and the angle ABC cannot be equal to the angle BAC by Art. 34; therefore the angle ABC must be greater than the angle BAC.

IV. RELATIONS BETWEEN THE TRIGONOMETRICAL FUNCTIONS OF THE SIDES AND ANGLES OF A SPHERICAL TRIANGLE.

37. *To express the cosine of an angle of a triangle in terms of sines and cosines of the sides.*



Let ABC be a spherical triangle, O the centre of the sphere. Let the tangent at A to the arc AC meet OC produced in E, and let the tangent at A to the arc AB meet OB produced in D; join ED. Thus the angle EAD is the angle A of the spherical triangle, and the angle EOD measures the side a.

From the triangles ADE and ODE we have

$$DE^2 = AD^2 + AE^2 - 2AD \,.\, AE \cos A,$$

$$DE^2 = OD^2 + OE^2 - 2OD \,.\, OE \cos a;$$

also the angles OAD and OAE are right angles, so that $OD^2 = OA^2 + AD^2$ and $OE^2 = OA^2 + AE^2$. Hence by subtraction we have

$$0 = 2OA^2 + 2AD \,.\, AE \cos A - 2OD \,.\, OE \cos a;$$

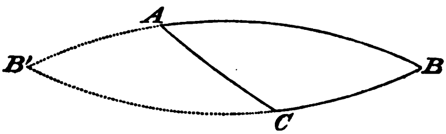
therefore, $$\cos a = \frac{OA}{OE} \cdot \frac{OA}{OD} + \frac{AE}{OE} \cdot \frac{AD}{OD} \cos A;$$

that is, $$\cos a = \cos b \cos c + \sin b \sin c \cos A.$$

Hence $\cos A$ is expressed as required.

38. We have supposed, in the construction of the preceding article, that the sides which contain the angle A are less than quadrants, for we have assumed that the tangents at A meet OB and OC respectively produced. We must now shew that the formula obtained is true when these sides are not less than quadrants. This we shall do by special examination of the cases in which one side or each side is greater than a quadrant or equal to a quadrant.

(1) Suppose only one of the sides greater than a quadrant, for example, AB. Produce BA and BC to meet in B'; and put $AB' = c'$, $CB' = a'$.



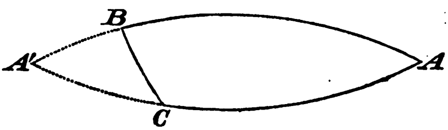
Then we have from the triangle $AB'C$, by what has been already proved,

$$\cos a' = \cos b \cos c' + \sin b \sin c' \cos B'AC\ ;$$

but $a' = \pi - a$, $c' = \pi - c$, $B'AC = \pi - A$; thus

$$\cos a = \cos b \cos c + \sin b \sin c \cos A.$$

(2) Suppose both the sides which contain the angle A to be greater than quadrants. Produce AB and AC to meet in A'; put $A'B = c'$, $A'C = b'$; then from the triangle $A'BC$, as before,

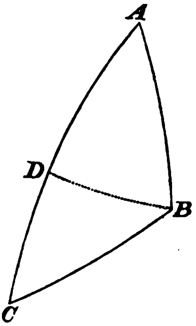


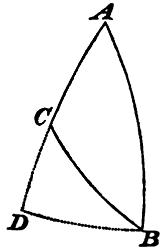
$$\cos a = \cos b' \cos c' + \sin b' \sin c' \cos A';$$

but $b' = \pi - b$, $c' = \pi - c$, $A' = A$; thus,

$$\cos a = \cos b \cos c + \sin b \sin c \cos A.$$

(3) Suppose that one of the sides which contain the angle A is a quadrant, for example, AB; on AC, produced if necessary,





take AD equal to a quadrant and draw BD. If BD is a quadrant B is a pole of AC (Art. 11); in this case $a = \frac{\pi}{2}$ and $A = \frac{\pi}{2}$ as well as $c = \frac{\pi}{2}$. Thus the formula to be verified reduces to the identity $0 = 0$. If BD be not a quadrant, the triangle BDC gives

$$\cos a = \cos CD \cos BD + \sin CD \sin BD \cos CDB,$$

and $\cos CDB = 0$, $\cos CD = \cos\left(\frac{\pi}{2} \sim b\right) = \sin b$, $\cos BD = \cos A$;

thus
$$\cos a = \sin b \cos A;$$

and this is what the formula in Art. 37 becomes when $c = \frac{\pi}{2}$.

(4) Suppose that both the sides which contain the angle A are quadrants. The formula then becomes $\cos a = \cos A$; and this is obviously true, for A is now the pole of BC, and thus $A = a$.

Thus the formula in Art. 37 is proved to be universally true.

39. The formula in Art. 37 may be applied to express the cosine of any angle of a triangle in terms of sines and cosines of the sides; thus we have the three formulæ,

$$\cos a = \cos b \cos c + \sin b \sin c \cos A,$$
$$\cos b = \cos c \cos a + \sin c \sin a \cos B,$$
$$\cos c = \cos a \cos b + \sin a \sin b \cos C.$$

These may be considered as the fundamental equations of Spherical Trigonometry; we shall proceed to deduce various formulæ from them.

40. *To express the sine of an angle of a spherical triangle.*

We have $\cos A = \dfrac{\cos a - \cos b \cos c}{\sin b \sin c}$;

therefore
$$\sin^2 A = 1 - \left(\frac{\cos a - \cos b \cos c}{\sin b \sin c}\right)^2$$
$$= \frac{(1-\cos^2 b)(1-\cos^2 c) - (\cos a - \cos b \cos c)^2}{\sin^2 b \sin^2 c}$$
$$= \frac{1 - \cos^2 a - \cos^2 b - \cos^2 c + 2\cos a \cos b \cos c}{\sin^2 b \sin^2 c};$$

therefore
$$\sin A = \frac{\sqrt{(1-\cos^2 a - \cos^2 b - \cos^2 c + 2\cos a \cos b \cos c)}}{\sin b \sin c}.$$

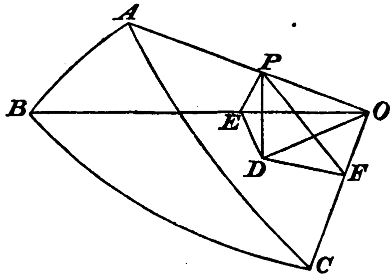
The radical on the right-hand side must be taken with the positive sign, because $\sin b$, $\sin c$, and $\sin A$ are all positive.

41. From the value of $\sin A$ in the preceding article it follows that
$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c},$$
for each of these is equal to the same expression, namely,
$$\frac{\sqrt{(1-\cos^2 a - \cos^2 b - \cos^2 c + 2\cos a \cos b \cos c)}}{\sin a \sin b \sin c}.$$

Thus *the sines of the angles of a spherical triangle are proportional to the sines of the opposite sides.* We will give an independent proof of this proposition in the following article.

42. *The sines of the angles of a spherical triangle are proportional to the sines of the opposite sides.*

Let ABC be a spherical triangle, O the centre of the sphere. Take any point P in OA, draw PD perpendicular to the plane



BOC, and from D draw DE, DF perpendicular to OB, OC respectively; join PE, PF, OD.

Since PD is perpendicular to the plane BOC, it is perpendicular to every line in that plane; hence

$$PE^2 = PD^2 + DE^2 = PO^2 - OD^2 + DE^2 = PO^2 - OE^2 ;$$

thus PEO is a right angle. Therefore $PE = OP \sin POE = OP \sin c$; and $PD = PE \sin PED = PE \sin B = OP \sin c \sin B$.

Similarly, $PD = OP \sin b \sin C$; therefore

$$OP \sin c \sin B = OP \sin b \sin C ;$$

therefore $$\frac{\sin B}{\sin C} = \frac{\sin b}{\sin c}.$$

The figure supposes b, c, B, and C all less than a right angle; it will be found on examination that the proof will hold when the figure is modified to meet any case which can occur. If, for instance, B alone is greater than a right angle, the point D will fall beyond OB instead of between OB and OC; then PED will be the *supplement* of B, and thus $\sin PED$ is still equal to $\sin B$.

43. *To prove that* $\cot a \sin b = \cot A \sin C + \cos b \cos C$.

We have $$\cos a = \cos b \cos c + \sin b \sin c \cos A,$$

$$\cos c = \cos a \cos b + \sin a \sin b \cos C,$$

$$\sin c = \sin a \frac{\sin C}{\sin A}.$$

Substitute the values of $\cos c$ and $\sin c$ in the first equation; thus

$$\cos a = (\cos a \cos b + \sin a \sin b \cos C) \cos b + \frac{\sin a \sin b \cos A \sin C}{\sin A};$$

by transposition

$$\cos a \sin^2 b = \sin a \sin b \cos b \cos C + \sin a \sin b \cot A \sin C;$$

divide by $\sin a \sin b$; thus

$$\cot a \sin b = \cos b \cos C + \cot A \sin C.$$

44. By interchanging the letters five other formulæ may be obtained like that in the preceding article; the whole six formulæ will be as follows:

$$\cot a \sin b = \cot A \sin C + \cos b \cos C,$$
$$\cot b \sin a = \cot B \sin C + \cos a \cos C,$$
$$\cot b \sin c = \cot B \sin A + \cos c \cos A,$$
$$\cot c \sin b = \cot C \sin A + \cos b \cos A,$$
$$\cot c \sin a = \cot C \sin B + \cos a \cos B,$$
$$\cot a \sin c = \cot A \sin B + \cos c \cos B.$$

45. *To express the sine, cosine, and tangent, of half an angle of a triangle as functions of the sides.*

We have, by Art. 37, $\cos A = \dfrac{\cos a - \cos b \cos c}{\sin b \sin c}$;

therefore $1 - \cos A = 1 - \dfrac{\cos a - \cos b \cos c}{\sin b \sin c} = \dfrac{\cos (b-c) - \cos a}{\sin b \sin c}$;

therefore $$\sin^2 \frac{A}{2} = \frac{\sin \frac{1}{2}(a+b-c) \sin \frac{1}{2}(a-b+c)}{\sin b \sin c}.$$

Let $2s = a + b + c$, so that s is half the sum of the sides of the triangle; then

$$a + b - c = 2s - 2c = 2(s-c), \quad a - b + c = 2s - 2b = 2(s-b);$$

thus $$\sin^2 \frac{A}{2} = \frac{\sin (s-b) \sin (s-c)}{\sin b \sin c},$$

and $$\sin\frac{A}{2} = \sqrt{\left\{\frac{\sin(s-b)\sin(s-c)}{\sin b \sin c}\right\}}.$$

Also, $$1 + \cos A = 1 + \frac{\cos a - \cos b \cos c}{\sin b \sin c} = \frac{\cos a - \cos(b+c)}{\sin b \sin c};$$

therefore

$$\cos^2\frac{A}{2} = \frac{\sin\frac{1}{2}(a+b+c)\sin\frac{1}{2}(b+c-a)}{\sin b \sin c} = \frac{\sin s \sin(s-a)}{\sin b \sin c},$$

and $$\cos\frac{A}{2} = \sqrt{\left\{\frac{\sin s \sin(s-a)}{\sin b \sin c}\right\}}.$$

From the expressions for $\sin\frac{A}{2}$ and $\cos\frac{A}{2}$ we deduce

$$\tan\frac{A}{2} = \sqrt{\left\{\frac{\sin(s-b)\sin(s-c)}{\sin s \sin(s-a)}\right\}}.$$

The positive sign must be given to the radicals which occur in this article, because $\frac{A}{2}$ is less than a right angle, and therefore its sine, cosine, and tangent are all positive.

46. Since $\sin A = 2\sin\frac{A}{2}\cos\frac{A}{2}$, we obtain

$$\sin A = \frac{2}{\sin b \sin c}\{\sin s \sin(s-a)\sin(s-b)\sin(s-c)\}^{\frac{1}{2}}.$$

It may be shewn that the expression for $\sin A$ in Art. 40 agrees with the present by putting the numerator of that expression in factors, as in *Plane Trigonometry*, Art. 115. We shall find it convenient to use a symbol for the radical in the value of $\sin A$; we shall denote it by n, so that

$$n^2 = \sin s \sin(s-a)\sin(s-b)\sin(s-c),$$

and $$4n^2 = 1 - \cos^2 a - \cos^2 b - \cos^2 c + 2\cos a \cos b \cos c.$$

47. *To express the cosine of a side of a triangle in terms of sines and cosines of the angles.*

In the formula of Art. 37 we may, by Art. 28, change the sides into the supplements of the corresponding angles and the angle into the supplement of the corresponding side; thus

$$\cos(\pi - A) = \cos(\pi - B)\cos(\pi - C) + \sin(\pi - B)\sin(\pi - C)\cos(\pi - a),$$

that is, $\cos A = -\cos B\cos C + \sin B\sin C\cos a.$

Similarly $\cos B = -\cos C\cos A + \sin C\sin A\cos b,$

and $\cos C = -\cos A\cos B + \sin A\sin B\cos c.$

48. The formulæ in Art. 44 will of course remain true when the angles and sides are changed into the supplements of the corresponding sides and angles respectively; it will be found, however, that no *new* formulæ are thus obtained, but only the *same* formulæ over again. This consideration will furnish some assistance in retaining those formulæ accurately in the memory.

49. *To express the sine, cosine, and tangent, of half a side of a triangle as functions of the angles.*

We have, by Art. 47, $\cos a = \dfrac{\cos A + \cos B\cos C}{\sin B\sin C}$;

therefore

$$1 - \cos a = 1 - \frac{\cos A + \cos B\cos C}{\sin B\sin C} = -\frac{\cos A + \cos(B + C)}{\sin B\sin C};$$

therefore $\sin^2\dfrac{a}{2} = -\dfrac{\cos\frac{1}{2}(A + B + C)\cos\frac{1}{2}(B + C - A)}{\sin B\sin C}.$

Let $2S = A + B + C$; then $B + C - A = 2(S - A)$, therefore

$$\sin^2\frac{a}{2} = -\frac{\cos S\cos(S - A)}{\sin B\sin C},$$

and

$$\sin\frac{a}{2} = \sqrt{\left\{-\frac{\cos S\cos(S - A)}{\sin B\sin C}\right\}}.$$

Also $$1+\cos a = 1 + \frac{\cos A + \cos B \cos C}{\sin B \sin C} = \frac{\cos A + \cos (B-C)}{\sin B \sin C};$$

therefore

$$\cos^2 \frac{a}{2} = \frac{\cos \frac{1}{2}(A-B+C) \cos \frac{1}{2}(A+B-C)}{\sin B \sin C} = \frac{\cos (S-B) \cos (S-C)}{\sin B \sin C},$$

and $$\cos \frac{a}{2} = \sqrt{\left\{\frac{\cos (S-B) \cos (S-C)}{\sin B \sin C}\right\}}.$$

Hence $$\tan \frac{a}{2} = \sqrt{\left\{-\frac{\cos S \cos (S-A)}{\cos (S-B) \cos (S-C)}\right\}}.$$

The positive sign must be given to the radicals which occur in this article, because $\frac{a}{2}$ is less than a right angle.

50. The expressions in the preceding article may also be obtained immediately from those given in Art. 45 by means of Art. 28.

It may be remarked that the values of $\sin \frac{a}{2}$, $\cos \frac{a}{2}$, and $\tan \frac{a}{2}$ are *real.* For S is greater than one right angle and less than three right angles by Art. 32; therefore $\cos S$ is *negative.* And in the polar triangle any side is less than the sum of the other two; thus $\pi - A$ is less than $\pi - B + \pi - C$; therefore $B+C-A$ is less than π; therefore $S-A$ is less than $\frac{\pi}{2}$, and $B+C-A$ is algebraically greater than $-\pi$, so that $S-A$ is algebraically greater than $-\frac{\pi}{2}$; therefore $\cos(S-A)$ is *positive.* Similarly also $\cos(S-B)$ and $\cos(S-C)$ are positive. Hence the values of $\sin \frac{a}{2}$, $\cos \frac{a}{2}$, and $\tan \frac{a}{2}$ are real.

51. Since $\sin a = 2 \sin \frac{a}{2} \cos \frac{a}{2}$, we obtain

$$\sin a = \frac{2}{\sin B \sin C}\{-\cos S \cos (S-A) \cos (S-B) \cos (S-C)\}^{\frac{1}{2}}.$$

We shall use N for $\{-\cos S \cos (S-A) \cos (S-B) \cos (S-C)\}^{\frac{1}{2}}$.

52. *To prove Napier's*

We have $\frac{\sin A}{\sin a} =$

then, by a theorem of Algeb

$$m = \frac{\sin}{\sin}$$

and also $m = \frac{\sin}{\sin}$

Now $\cos A + \cos B \cos C = $ si

and $\cos B + \cos A \cos C = $ si

therefore, by addition,

$(\cos A + \cos B)(1 + \cos$

therefore by (1) we have

$$\frac{\sin A + \sin B}{\cos A + \cos B} =$$

that is, $\tan \frac{1}{2}(A + B)$

Similarly from (3) and (2

$$\frac{\sin A - \sin B}{\cos A + \cos B} =$$

that is, $\tan \frac{1}{2}(A - B)$

By writing $\pi - A$ for a,

$$\tan \frac{1}{2}(a + b) =$$

$$\tan \frac{1}{2}(a - b) =$$

The formulæ (4), (5), (6),
portions or analogies, and ar

Analogies; the last two may be proved without recurring to the polar triangle by starting with the formulæ in Art. 39.

53. In equation (4) of the preceding article, $\cos \frac{1}{2}(a-b)$ and $\cot \frac{C}{2}$ are necessarily positive quantities; hence the equation shews that $\tan \frac{1}{2}(A+B)$ and $\cos \frac{1}{2}(a+b)$ are of the same sign; thus $\frac{1}{2}(A+B)$ and $\frac{1}{2}(a+b)$ are either both less than a right angle or both greater than a right angle. This is expressed by saying that $\frac{1}{2}(A+B)$ and $\frac{1}{2}(a+b)$ *are of the same affection.*

54. *To prove Gauss's Theorems.*

We have $\cos c = \cos a \cos b + \sin a \sin b \cos C$; therefore,

$$1 + \cos c = 1 + \cos a \cos b + \sin a \sin b (\cos^2 \tfrac{1}{2} C - \sin^2 \tfrac{1}{2} C)$$
$$= \{1 + \cos(a-b)\} \cos^2 \tfrac{1}{2} C + \{1 + \cos(a+b)\} \sin^2 \tfrac{1}{2} C;$$

therefore $\cos^2 \frac{1}{2} c = \cos^2 \frac{1}{2}(a-b) \cos^2 \frac{1}{2} C + \cos^2 \frac{1}{2}(a+b) \sin^2 \frac{1}{2} C.$

Similarly, $\sin^2 \frac{1}{2} c = \sin^2 \frac{1}{2}(a-b) \cos^2 \frac{1}{2} C + \sin^2 \frac{1}{2}(a+b) \sin^2 \frac{1}{2} C.$

Now add unity to the square of each member of Napier's first two analogies; hence by the formulæ just proved

$$\sec^2 \tfrac{1}{2}(A+B) = \frac{\cos^2 \frac{1}{2} c}{\cos^2 \frac{1}{2}(a+b) \sin^2 \frac{1}{2} C},$$

$$\sec^2 \tfrac{1}{2}(A-B) = \frac{\sin^2 \frac{1}{2} c}{\sin^2 \frac{1}{2}(a+b) \sin^2 \frac{1}{2} C}.$$

Extract the square roots; thus, since $\frac{1}{2}(A+B)$ and $\frac{1}{2}(a+b)$ are of the same affection, we obtain

$$\cos \tfrac{1}{2}(A+B) \cos \tfrac{1}{2} c = \cos \tfrac{1}{2}(a+b) \sin \tfrac{1}{2} C \quad \text{..........(1)},$$

$$\cos \tfrac{1}{2}(A-B) \sin \tfrac{1}{2} c = \sin \tfrac{1}{2}(a+b) \sin \tfrac{1}{2} C \quad \text{..........(2)}.$$

Multiply the first two of Napier's analogies respectively by these results; thus

$$\sin \tfrac{1}{2}(A+B) \cos \tfrac{1}{2} c = \cos \tfrac{1}{2}(a-b) \cos \tfrac{1}{2} C \quad \text{..........(3)},$$

$$\sin \tfrac{1}{2}(A-B) \sin \tfrac{1}{2} c = \sin \tfrac{1}{2}(a-b) \cos \tfrac{1}{2} C \quad \text{..........(4)}.$$

The last four formulæ are called *Gauss's Theorems*, although they are really due to Delambre.

55. The properties of supplemental triangles were proved geometrically in Art. 27, and by means of these properties the formulæ in Art. 47 were obtained; but these formulæ may be deduced analytically from those in Art. 39, and thus the whole subject may be made to depend upon the formulæ of Art. 39.

For from Art. 39 we obtain expressions for $\cos A$, $\cos B$, $\cos C$; and from these we find

$$\cos A + \cos B \cos C$$
$$= \frac{(\cos a - \cos b \cos c) \sin^2 a + (\cos b - \cos a \cos c)(\cos c - \cos a \cos b)}{\sin^2 a \sin b \sin c}.$$

In the numerator of this fraction write $1 - \cos^2 a$ for $\sin^2 a$; thus the numerator will be found to reduce to

$$\cos a\,(1 - \cos^2 a - \cos^2 b - \cos^2 c + 2 \cos a \cos b \cos c)$$

and this is equal to $\cos a \sin B \sin C \sin^2 a \sin b \sin c$, (Art. 41);

therefore $\qquad \cos A + \cos B \cos C = \cos a \sin B \sin C.$

Similarly the other two corresponding formulæ may be proved.

Thus the formulæ in Art. 47 are established; and therefore, without assuming the existence and properties of the Polar Triangle, we deduce the following theorem:—*If the sides and angles of a spherical triangle be changed respectively into the supplements of the corresponding angles and sides the fundamental formulæ of Art.* 39 *hold good, and therefore also all results deducible from them.*

56. The formulæ in the present chapter may be applied to establish analytically various propositions respecting spherical triangles which either have been proved geometrically in the preceding chapter, or may be so proved. Thus, for example, the second of Napier's analogies is

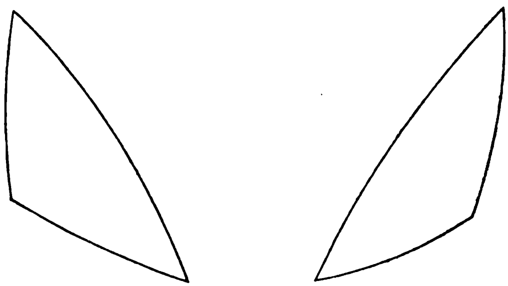
$$\tan \tfrac{1}{2}(A - B) = \frac{\sin \frac{1}{2}(a - b)}{\sin \frac{1}{2}(a + b)} \cot \frac{C}{2};$$

this shews that $\frac{1}{2}(A-B)$ is positive, negative, or zero, according as $\frac{1}{2}(a-b)$ is positive, negative, or zero; thus we obtain all the results included in Arts. 33—36.

57. *If two triangles have two sides of the one equal to two sides of the other, each to each, and likewise the included angles equal, then their other angles will be equal, each to each, and likewise their bases will be equal.*

We may shew that the bases are equal by applying the first formula in Art. 39 to each triangle, supposing b, c, and A the same in the two triangles; then the remaining two formulæ of Art. 39 will shew that B and C are the same in the two triangles.

It should be observed that the two triangles in this case are *not* necessarily such that one may be made to *coincide with the other by superposition.* The sides of one may be equal to those of the other, each to each, but in a reverse order, as in the following figures.



Two triangles which are equal in this manner are said to be *symmetrically* equal; when they are equal so as to admit of superposition they are said to be *absolutely* equal.

58. *If two spherical triangles have two sides of the one equal to two sides of the other, each to each, but the angle which is contained by the two sides of the one greater than the angle which is contained by the two sides which are equal to them of the other, the base of that*

which has the greater angle will be greater than the base of the other; and conversely.

Let b and c denote the sides which are equal in the two triangles; let a be the base and A the opposite angle of one triangle, and a' and A' similar quantities for the other. Then

$$\cos a = \cos b \cos c + \sin b \sin c \cos A,$$

$$\cos a' = \cos b \cos c + \sin b \sin c \cos A';$$

therefore $$\cos a - \cos a' = \sin b \sin c (\cos A - \cos A');$$

that is,

$$\sin \tfrac{1}{2}(a + a') \sin \tfrac{1}{2}(a - a') = \sin b \sin c \sin \tfrac{1}{2}(A + A') \sin \tfrac{1}{2}(A - A');$$

this shews that $\frac{1}{2}(a - a')$ and $\frac{1}{2}(A - A')$ are of the same sign.

59. *If on a sphere any point be taken within a circle which is not its pole, of all the arcs which can be drawn from that point to the circumference, the greatest is that in which the pole is, and the other part of that produced is the least; and of any others, that which is nearer to the greatest is always greater than one more remote; and from the same point to the circumference there can be drawn only two arcs which are equal to each other, and these make equal angles with the shortest arc on opposite sides of it.*

This follows readily from the preceding three articles.

60. We will give another proof of the fundamental formulæ in Art. 39, which is very simple, requiring only a knowledge of the elements of Co-ordinate Geometry.

Suppose ABC any spherical triangle, O the centre of the sphere; take O as the origin of co-ordinates, and let the axis of z pass through C. Let x_1, y_1, z_1 be the co-ordinates of A, and x_2, y_2, z_2 those of B, r the radius of the sphere. Then the square of the straight line AB is equal to

$$(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2,$$

and also to $$r^2 + r^2 - 2r^2 \cos AOB;$$

and $x_1^2+y_1^2+z_1^2=r^2$, $x_2^2+y_2^2+z_2^2=r^2$; thus

$$x_1x_2+y_1y_2+z_1z_2=r^2\cos AOB.$$

Now make the usual substitutions in passing from rectangular to polar co-ordinates, namely,

$$z_1=r\cos\theta_1,\quad x_1=r\sin\theta_1\cos\phi_1,\quad y_1=r\sin\theta_1\sin\phi_1,$$
$$z_2=r\cos\theta_2,\quad x_2=r\sin\theta_2\cos\phi_2,\quad y_2=r\sin\theta_2\sin\phi_2;$$

thus we obtain

$$\cos\theta_2\cos\theta_1+\sin\theta_2\sin\theta_1\cos(\phi_1-\phi_2)=\cos AOB,$$

that is, in the ordinary notation of Spherical Trigonometry,

$$\cos a\cos b+\sin a\sin b\cos C=\cos c.$$

This method has the advantage of giving a *perfectly general proof*, as all the equations used are universally true.

EXAMPLES.

1. If $A=a$, shew that B and b are equal or supplemental, as also C and c.

2. If one angle of a triangle be equal to the sum of the other two, the greatest side is double of the distance of its middle point from the opposite angle.

3. When does the polar triangle coincide with the primitive triangle?

4. If D be the middle point of AB, shew that

$$\cos AC+\cos BC=2\cos\tfrac{1}{2}AB\cos CD.$$

5. If two angles of a spherical triangle be respectively equal to the sides opposite to them, prove that the remaining side is the supplement of the remaining angle; or else that the triangle has two quadrants and two right angles, and then the remaining side is equal to the remaining angle.

6. In an equilateral triangle, prove that $2\cos\frac{a}{2}\sin\frac{A}{2}=1$.

7. In an equilateral triangle, prove that $\tan^2\frac{a}{2}=1-2\cos A$; hence deduce the limits between which the sides and angles of an equilateral triangle are restricted.

8. In an equilateral triangle, prove that $\sec A = 1 + \sec a$.

9. If the three sides of a spherical triangle be halved and a new triangle formed, the angle θ between the new sides $\frac{b}{2}$ and $\frac{c}{2}$ is given by $\cos\theta = \cos A + \frac{1}{2}\tan\frac{b}{2}\tan\frac{c}{2}\sin^2\theta$.

10. AB, CD are quadrants on the surface of a sphere intersecting in E, the extremities being joined by great circles; prove that

$$\cos AEC = \cos AC \cos BD - \cos BC \cos AD.$$

11. If $b + c = \pi$, prove that $\sin 2B + \sin 2C = 0$.

12. If DE be an arc of a great circle bisecting the sides AB, AC of a spherical triangle in D and E, P a pole of DE, and PB, PD, PE, PC be joined by arcs of great circles, shew that the angle BPC = twice the angle DPE.

13. Shew that

$$\sin b \sin c + \cos b \cos c \cos A = \sin B \sin C - \cos B \cos C \cos a.$$

14. If D be any point in the side BC of a triangle, shew that

$$\cos AD \sin BC = \cos AB \sin DC + \cos AC \sin BD.$$

15. Prove that if θ, ϕ, ψ be the lengths of arcs of great circles drawn from A, B, C perpendicular to the opposite sides,

$$\sin a \sin\theta = \sin b \sin\phi = \sin c \sin\psi$$
$$= \sqrt{(1-\cos^2 a - \cos^2 b - \cos^2 c + 2\cos a\cos b\cos c)}.$$

16. In a spherical triangle, if θ, ϕ, ψ be the arcs bisecting the angles A, B, C respectively and terminated by the opposite sides, shew that

$$\cot\theta\cos\frac{A}{2}+\cot\phi\cos\frac{B}{2}+\cot\psi\cos\frac{C}{2}=\cot a+\cot b+\cot c.$$

17. Two ports are in the same parallel of latitude, their common latitude being l and their difference of longitude 2λ; shew that the saving of distance in sailing from one to the other on the great circle, instead of sailing due East or West, is

$$2r\{\lambda\cos l-\sin^{-1}(\sin\lambda\cos l)\},$$

λ being expressed in circular measure, and r being the radius of the Earth.

18. If a ship be proceeding uniformly along a great circle and the observed latitudes be l_1, l_2, l_3, at equal intervals of time, in each of which the distance traversed is s, shew that

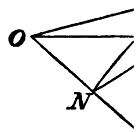
$$s=r\cos^{-1}\frac{\sin\frac{1}{2}(l_1+l_3)\cos\frac{1}{2}(l_1-l_3)}{\sin l_2},$$

r denoting the Earth's radius; and shew that the change of longitude may also be found in terms of the three latitudes.

V. SOLUTION OF RIGHT-ANGLED TRIANGLES.

61. In every spherical triangle there are six elements, namely, the three sides and the three angles, besides the radius of the sphere, which is supposed constant. The solution of spherical triangles is the process by which, when the values of a sufficient number of the six elements are given, we calculate the values of the remaining elements. It will appear, as we proceed, that when the values of three of the elements are given, those of the remaining three can generally be found. We begin with the right-angled triangle; here two elements, in addition to the right angle, will be supposed known.

62. The formulæ requis
triangles may be obtained f
posing one of the angles a r
may also be obtained very e
we will now shew.



Let ABC be a spherical
let O be the centre of the sph
PM perpendicular to OC, and
OB, and join PN. Then PM
plane AOC is perpendicular to

$$PN^2 = PM^2 + MN^2 = OP^2 -$$

therefore PNO is a right angl

$$\frac{ON}{OP} = \frac{ON}{OM} \cdot \frac{OM}{OP}, \text{ that}$$

$$\frac{PM}{OP} = \frac{PM}{PN} \cdot \frac{PN}{OP}, \text{ that}$$

Similarly

$$\frac{MN}{ON} = \frac{MN}{PN} \cdot \frac{PN}{ON}, \text{ that}$$

Similarly

$$\frac{PM}{OM} = \frac{PM}{MN} \cdot \frac{MN}{OM}, \text{ that i}$$

Similarly

Multiply together the two formulæ (4); thus,

$$\tan A \tan B = \frac{\tan a \tan b}{\sin a \sin b} = \frac{1}{\cos a \cos b} = \frac{1}{\cos c} \text{ by (1);}$$

therefore $$\cos c = \cot A \cot B \quad \text{.......................} \quad (5).$$

Multiply crosswise the second formula in (2) and the first in (3); thus $\sin a \cos B \tan c = \tan a \sin A \sin c$;

therefore $$\cos B = \frac{\sin A \cos c}{\cos a} = \sin A \cos b \text{ by (1).}$$

thus $$\cos B = \sin A \cos b$$
Similarly $$\cos A = \sin B \cos a \quad \text{..........................} \quad (6).$$

These six formulæ comprise ten equations; and thus we can solve every case of right-angled triangles. For every one of these ten equations is a distinct combination involving three out of the five quantities a, b, c, A, B; and out of five quantities only ten combinations of three can be formed. Thus any two of the five quantities being given and a third required, some one of the preceding ten equations will serve to determine that third quantity.

63. As we have stated, the above six formulæ may be obtained from those given in the preceding chapter by supposing C a right angle. Thus (1) follows from Art. 39, (2) from Art. 41, (3) from the fourth and fifth equations of Art. 44, (4) from the first and second equations of Art. 44, (5) from the third equation of Art. 47, (6) from the first and second equations of Art. 47.

Since the six formulæ may be obtained from those given in the preceding chapter which have been proved to be universally true, we do not stop to shew that the demonstration of Art. 62 may be applied to every case which can occur; the student may for exercise investigate the modifications which will be necessary when we suppose one or more of the quantities a, b, c, A, B equal to a right angle or greater than a right angle.

64. Certain properties of right-angled triangles are deducible from the formulæ of Art. 62.

From (1) it follows that cos c has the same sign as the product cos a cos b; hence either all the cosines are positive, or else only one is positive. Therefore *in a right-angled triangle either all the three sides are less than quadrants, or else one side is less than a quadrant and the other two sides are greater than quadrants.*

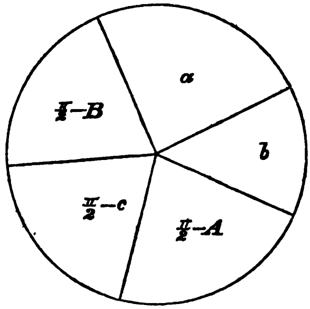
From (4) it follows that tan a has the same sign as tan A. Therefore A and a are either both greater than $\frac{\pi}{2}$, or both less than $\frac{\pi}{2}$; this is expressed by saying that A and a are of the *same affection.* Similarly B and b are of the same affection.

65. The formulæ of Art. 62 are comprised in the following enunciations, which the student will find it useful to remember; the results are distinguished by the same numbers as have been already applied to them in Art. 62; the side opposite the right angle is called the *hypothenuse:*

Cos hyp = product of cosines of sides ....(1),
Cos hyp = product of cotangents of angles ....(5),
Sine side = sine of opposite angle × sine hyp............(2),
Tan side = tan hyp × cos included angle ....(3),
Tan side = tan opposite angle × sine of other side......(4),
Cos angle = cos opposite side × sine of other angle......(6).

66. *Napier's Rules.* The formulæ of Art. 62 are comprised in two rules, which are called, from their inventor, *Napier's Rules of Circular Parts.* Napier was also the inventor of Logarithms, and the Rules of Circular Parts were first published by him in a work entitled *Mirifici Logarithmorum Canonis Descriptio*...... Edinburgh, 1614. These rules we will now explain.

The right angle is left out of consideration; the two sides which include the right angle, the complement of the hypothenuse, and the complements of the other angles are called the *circular*



parts of the triangle. Thus there are *five* circular parts, namely, a, b, $\frac{\pi}{2}-A$, $\frac{\pi}{2}-c$, $\frac{\pi}{2}-B$; and these are supposed to be ranged round a circle in the order in which they naturally occur with respect to the triangle.

Any one of the five parts may be selected and called the *middle part*, then the two parts next to it are called *adjacent parts*, and the remaining two parts are called *opposite parts*. For example, if $\frac{\pi}{2}-B$ is selected as the middle part, then the adjacent parts are a and $\frac{\pi}{2}-c$, and the opposite parts are b and $\frac{\pi}{2}-A$.

Then Napier's Rules are the following:

sine of the middle part = product of tangents of adjacent parts,

sine of the middle part = product of cosines of opposite parts.

67. Napier's Rules may be proved by shewing that they agree with the results already established. The following table shews the required agreement: in the first column are given the *middle parts*, in the second column the results of Napier's Rules, and in the third column the same results expressed as in Art. 62, with the number for reference used in that article.

$\frac{\pi}{2} - c \quad \sin\left(\frac{\pi}{2} - c\right) = \tan\left(\frac{\pi}{2} -\right.$

$\sin\left(\frac{\pi}{2} - c\right) = \cos a \ \text{co}$

$\frac{\pi}{2} - B \quad \sin\left(\frac{\pi}{2} - B\right) = \tan a \ \text{ta}$

$\sin\left(\frac{\pi}{2} - B\right) = \cos b \ \text{cc}$

$a \quad \sin a = \tan b \ \text{ta}$

$\sin a = \cos\left(\frac{\pi}{2} -\right.$

$b \quad \sin b = \tan\left(\frac{\pi}{2} -\right.$

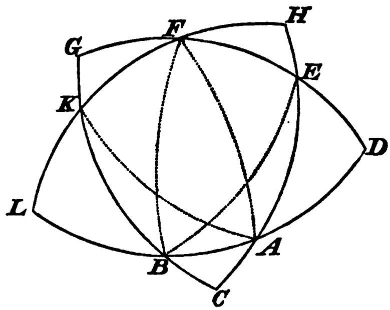
$\sin b = \cos\left(\frac{\pi}{2} -\right.$

$\frac{\pi}{2} - A \quad \sin\left(\frac{\pi}{2} - A\right) = \tan b \ \text{ta}$

$\sin\left(\frac{\pi}{2} - A\right) = \cos a \ \text{co}$

The last four cases need
that they are only repetitio
the seventh and eighth are
the ninth and tenth of the t

68. It has been some
preceding article is the onl
proved; this statement, hov
method Napier himself indi
Mirifici Logarithmorum Ca
will now briefly explain:



Let ABC be a spherical triangle right-angled at C; with as pole describe a great circle $DEFG$, and with A as pole descril a great circle $HFKL$, and produce the sides of the original triang ABC to meet these great circles. Then since B is a pole of DEF the angles at D and G are right angles, and since A is a pole $HFKL$ the angles at H and L are right angles. Hence the fi triangles BAC, AED, EFH, FKG, KBL are all *right-angled;* ar moreover it will be found on examination that, although the el ments of these triangles are different, yet *their circular parts a the same.* We will consider, for example, the triangle AED; tl angle EAD is equal to the angle BAC; the side AD is the cor plement of AB; as the angles at C and G are right angles E is pole of GC (Art. 13), therefore EA is the complement of AC; B is a pole of DE the angle BED is a right angle, therefore tl angle AED is the complement of the angle BEC, that is, tl angle AED is the complement of the side BC (Art. 12); and sim larly the side DE is equal to the angle DBE, and is therefore tl complement of the angle ABC. Hence, if we denote the elemen of the triangle ABC as usual by a, b, c, A, B, we have in tl triangle AED the hypothenuse equal to $\frac{\pi}{2}-b$, the angles equal A and $\frac{\pi}{2}-a$, and the sides respectively opposite these angles equ to $\frac{\pi}{2}-B$ and $\frac{\pi}{2}-c$. The *circular parts* of AED are therefore tl same as those of ABC. Similarly the remaining three of the five right-angled triangles may be shewn to have the same circular parts as the triangle ABC has.

Now take *two* of the theorems in Art. 65, for example (1) and (3); then the truth of the *ten* cases comprised in Napier's Rules will be found to follow from applying the two theorems in succession to the five triangles formed in the preceding figure. Thus this method of considering Napier's Rules regards each Rule, not as the statement of dissimilar properties of one triangle, but as the statement of similar properties of five allied triangles.

69. In Napier's work a figure is given of which that in the preceding Article is a copy, except that different letters are used; Napier briefly intimates that the truth of the Rules can be easily seen by means of this figure, as well as by the method of induction from consideration of all the cases which can occur. The late T. S. Davies, in his edition of Dr Hutton's *Course of Mathematics*, drew attention to Napier's own views and expanded the demonstration by a systematic examination of the figure of the preceding article. It is however easy to evade the necessity of examining the whole figure; all that is wanted is to observe the connexion between the triangle AED and the triangle BAC. For let a_1, a_2, a_3, a_4, a_5 represent the elements of the triangle BAC taken in order, beginning with the hypothenuse and omitting the right angle; then the elements of the triangle AED taken in order, beginning with the hypothenuse and omitting the right angle, are $\frac{\pi}{2}-a_3$, $\frac{\pi}{2}-a_4$, $\frac{\pi}{2}-a_5$, $\frac{\pi}{2}-a_1$, and a_2. If, therefore, to characterise the former we introduce a new set of quantities p_1, p_2, p_3, p_4, p_5, such that $a_1+p_1=a_2+p_2=a_5+p_5=\frac{\pi}{2}$, and that $p_3=a_3$ and $p_4=a_4$, then the original triangle being characterised by p_1, p_2, p_3, p_4, p_5, the second triangle will be characterised by p_3, p_4, p_5, p_1, p_2. As the second triangle can give rise to a third in like manner, and so on, we see that every right-angled triangle is one of a system of five such triangles which are all characterised by the quantities

p_1, p_2, p_3, p_4, p_5, always taken in order, each quantity in its turn standing first.

70. Opinions have differed with respect to the *utility* of Napier's Rules in practice. Thus Woodhouse says, "In the whole compass of mathematical science there cannot be found, perhaps, rules which more completely attain that which is the proper object of rules, namely, facility and brevity of computation." (*Trigonometry*, chap. x.) On the other hand may be set the following sentence from Airy's Trigonometry (*Encyclopædia Metropolitana*): "In the opinion of Delambre (and no one was better qualified by experience to give an opinion) these theorems are best recollected by the practical calculator in their unconnected form." Professor De Morgan strongly objects to Napier's Rules, and says (*Spherical Trigonometry*, Art. 17): "There are certain mnemonical formulæ called *Napier's Rules of Circular Parts*, which are generally explained. We do not give them, because we are convinced that they only create confusion instead of assisting the memory."

71. We shall now proceed to apply the formulæ of Art. 62 to the solution of right-angled triangles. We shall assume that the given quantities are subject to the limitations of Arts. 22 and 23, that is, a given side must be less than the semicircumference of a great circle, and a given angle less than two right angles. There will be six cases to consider.

72. *Having given the hypothenuse* c *and an angle* A.

Here we have from (3), (5) and (2) of Art. 62,

$$\tan b = \tan c \cos A, \quad \cot B = \cos c \tan A, \quad \sin a = \sin c \sin A.$$

Thus b and B are determined immediately without ambiguity; and as a must be of the same affection as A (Art. 64), a also is determined without ambiguity.

It is obvious from the formulæ of solution, that in this case the triangle is always possible.

If c and A are both right angles, a is a right angle, and b and B are indeterminate.

73. *Having given a side* t

Here we have from (3), (4)

$$\tan c = \frac{\tan b}{\cos A}, \quad \tan a = t\text{a}$$

Here c, a, B are determin
angle is always possible.

74. *Having given the two*

Here we have from (1) and

$$\cos c = \cos a \cos b, \; \cot A$$

Here c, A, B are determin
angle is always possible.

75. *Having given the hyp*

Here we have from (1), (3)

$$\cos b = \frac{\cos c}{\cos a}, \quad \cos B$$

Here b, B, A are determin
be of the same affection as a.
that there are limitations of th
triangle; in fact, c must lie be
values found for $\cos b$, $\cos B$,
greater than unity.

If c and a are right angles,
indeterminate.

76. *Having given the two*

Here we have from (5) and

$$\cos c = \cot A \cot B, \quad \text{co}$$

Here c, a, b are determin
limitations of the data in order

suppose A less than $\frac{\pi}{2}$, then B must lie between $\frac{\pi}{2} - A$ and $\frac{\pi}{2} + A$; next suppose A greater than $\frac{\pi}{2}$, then B must lie between $\frac{\pi}{2} - (\pi - A)$ and $\frac{\pi}{2} + (\pi - A)$, that is, between $A - \frac{\pi}{2}$ and $\frac{3\pi}{2} - A$.

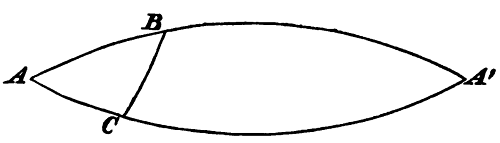
77. *Having given a side* a *and the opposite angle* A.

Here we have from (2), (4) and (6) of Art. 62,

$$\sin c = \frac{\sin a}{\sin A}, \quad \sin b = \tan a \cot A, \quad \sin B = \frac{\cos A}{\cos a}.$$

Here there is an ambiguity, as the parts are determined from their sines. If $\sin a$ be less than $\sin A$, there are two values admissible for c; corresponding to each of these there will be *in general* only one admissible value of b, since we must have $\cos c = \cos a \cos b$, and only one admissible value of B, since we must have $\cos c = \cot A \cot B$. Thus if one triangle exists with the given parts, there will be *in general* two, and only two, triangles with the given parts. We say *in general* in the preceding sentences, because if $a = A$ there will be only *one* triangle, unless a and A are each right angles, and then b and B become indeterminate.

It is easy to see from a figure that the ambiguity must occur in general.



For, suppose BAC to be a triangle which satisfies the given conditions; produce AB and AC to meet again in A'; then the triangle $A'BC$ also satisfies the given conditions, for it has a right angle at C, BC the given side, and $A' = A$ the given angle.

If $a = A$, then the formulæ of solution shew that c, b, and B are right angles; in this case A is the pole of BC, and the triangle $A'BC$ is symmetrically equal to the triangle ABC (Art. 57).

If a and A are both right angles, B is the pole of AC; B and b are then equal, but may have any value whatever.

There are limitations of the data in order to insure a possible triangle. A and a must have the same affection by Art. 64; hence the formulæ of solution shew that a must be less than A if both are acute, and greater than A if both are obtuse.

EXAMPLES.

If ABC be a triangle in which the angle C is a right angle, prove the following relations contained in Examples 1 to 5.

1. $\text{Sin}^2 \frac{c}{2} = \sin^2 \frac{a}{2} \cos^2 \frac{b}{2} + \cos^2 \frac{a}{2} \sin^2 \frac{b}{2}$.

2. $\text{Tan} \frac{1}{2}(c+a) \tan \frac{1}{2}(c-a) = \tan^2 \frac{b}{2}$.

3. $\text{Sin}(c-b) = \tan^2 \frac{A}{2} \sin(c+b)$.

4. $\text{Sin}\, a \tan \frac{1}{2} A - \sin b \tan \frac{1}{2} B = \sin(a-b)$.

5. $$\text{Sin}(c-a) = \sin b \cos a \tan \tfrac{1}{2} B,$$
$$\text{Sin}(c-a) = \tan b \cos c \tan \tfrac{1}{2} B.$$

6. If ABC be a spherical triangle, right-angled at C, and $\cos A = \cos^2 a$, shew that if A be not a right angle $b + c = \frac{1}{2}\pi$ or $\frac{3}{2}\pi$, according as b and c are both less or both greater than $\frac{\pi}{2}$.

7. If α, β be the arcs drawn from the right angle respectively perpendicular to and bisecting the hypothenuse c, shew that

$$\sin \frac{c}{2} \cdot \sqrt{(1 + \sin^2 \alpha)} = \sin \beta.$$

8. In a triangle, if C be a right angle and D the middle point of AB, shew that

$$4\cos^2\frac{c}{2}\sin^2 CD = \sin^2 a + \sin^2 b.$$

9. In a right-angled triangle, if δ be the length of the arc drawn from C at right angles to the hypothenuse AB, shew that

$$\cot\delta = \sqrt{(\cot^2 a + \cot^2 b)}.$$

10. OAA_1 is a spherical triangle right-angled at A_1 and acute-angled at A; the arc $A_1 A_2$ of a great circle is drawn perpendicular to OA, $A_2 A_3$ is drawn perpendicular to OA_1, and so on; prove that $A_n A_{n+1}$ vanishes when n becomes infinite; and find the value of $\cos AA_1 \cos A_1A_2 \cos A_2A_3$......to infinity.

11. ABC is a right-angled spherical triangle, A not being the right angle; prove that if $A = a$, then c and b are quadrants.

12. If δ be the length of the arc drawn from C at right angles to AB in *any* triangle, shew that

$$\cos\delta = \operatorname{cosec} c\,(\cos^2 a + \cos^2 b - 2\cos a\cos b\cos c)^{\frac{1}{2}}.$$

13. ABC is a great circle of a sphere; AA', BB', CC', are arcs of great circles drawn perpendicular to ABC and reckoned positive when they lie on the same side of it; prove that the condition of A', B', C' lying in a great circle is

$$\tan AA'\sin BC + \tan BB'\sin CA + \tan CC'\sin AB = 0.$$

14. Perpendiculars are drawn from the angles A, B, C of any triangle meeting the opposite sides in D, E, F respectively; shew that

$$\tan BD\tan CE\tan AF = \tan DC\tan EA\tan FB.$$

15. Ox, Oy are two great circles of a sphere at right angles to each other, P any point in AB another great circle. $OC = p$ is the arc perpendicular to AB from O, making the angle $COx = \alpha$ with

'x. PM, PN are arcs perpendicular to Ox, Oy respectively; shew hat if $OM = x$ and $ON = y$,

$$\cos a \tan x + \sin a \tan y = \tan p.$$

16. The position of a point on a sphere, with reference to two reat circles at right angles to each other as axes, is determined y the portions θ, ϕ of these circles cut off by great circles through he point, and through two points on the axes, each $\frac{\pi}{2}$ from their oint of intersection; shew that if the three points (θ, ϕ), (θ', ϕ'), θ'', $\phi'')$ lie on the same great circle

$$\tan \phi (\tan \theta' - \tan \theta'') + \tan \phi' (\tan \theta'' - \tan \theta)$$
$$+ \tan \phi'' (\tan \theta - \tan \theta') = 0.$$

17. If a point on a sphere be referred to two great circles at ight angles to each other as axes, by means of the portions of hese axes cut off by great circles drawn through the point and wo points on the axes each 90° from their intersection, shew that he equation to a great circle is

$$\tan \theta \cot a + \tan \phi \cot \beta = 1.$$

18. In a triangle, if $A = \frac{\pi}{5}$, $B = \frac{\pi}{3}$, and $C = \frac{\pi}{2}$, shew that $\iota + b + c = \frac{\pi}{2}$.

VI. SOLUTION OF OBLIQUE-ANGLED TRIANGLES.

78. The solution of oblique-angled triangles may be made in some cases to depend immediately upon the solution of right-angled triangles; we will indicate these cases before considering the subject generally.

(1) Suppose a triangle to have one of its given sides equal to a *quadrant*. In this case the polar triangle has its corresponding angle a right angle; the polar triangle can therefore be solved by the rules of the preceding chapter, and thus the elements of the primitive triangle become known.

(2) Suppose among the given elements of a triangle there are two *equal sides* or two *equal angles*. By drawing an arc from the vertex to the middle point of the base, the triangle is divided into two equal *right-angled* triangles; by the solution of one of these right-angled triangles the required elements can be found.

(3) Suppose among the given elements of a triangle there are two sides, one of which is the supplement of the other, or two angles, one of which is the supplement of the other. Suppose, for example, that $b + c = \pi$, or else that $B + C = \pi$; then produce BA and BC to meet in B' (see the first figure to Art. 38); then the triangle $B'AC$ has two equal sides given, or else two equal angles given; and by the preceding case the solution of it can be made to depend upon that of a right-angled triangle.

79. We now proceed to the solution of oblique-angled triangles in general. There will be six cases to consider.

80. *Having given the three sides.*

Here we have $\cos A = \dfrac{\cos a - \cos b \cos c}{\sin b \sin c}$, and similar formulæ for $\cos B$ and $\cos C$. Or if we wish to use formulæ suited to logarithms, we may take the formula for the sine, cosine, or tangent of half an angle given in Art. 45. In selecting a formula, attention should be paid to the remarks in *Plane Trigonometry*, Chap. XII. towards the end.

81. *Having given the three angles.*

Here we have $\cos a = \dfrac{\cos A + \cos B \cos C}{\sin B \sin C}$, and similar formulæ for $\cos b$ and $\cos c$. Or if we wish to use formulæ suited to logarithms, we may take the formula for the sine, cosine, or tangent of half a side given in Art. 49.

There is no ambiguity in the two preceding cases; the triangles however may be impossible with the given elements.

82. *Having given two sides and the included angle* (a, C, b).

By Napier's analogies

$$\tan \tfrac{1}{2}(A + B) = \frac{\cos \frac{1}{2}(a - b)}{\cos \frac{1}{2}(a + b)} \cot \tfrac{1}{2} C,$$

$$\tan \tfrac{1}{2}(A - B) = \frac{\sin \frac{1}{2}(a - b)}{\sin \frac{1}{2}(a + b)} \cot \tfrac{1}{2} C;$$

these determine $\frac{1}{2}(A + B)$ and $\frac{1}{2}(A - B)$, and thence A and B.

Then c may be found from the formula $\sin c = \dfrac{\sin a \sin C}{\sin A}$; in this case, since c is found from its sine, it may be uncertain which of two values is to be given to it; the point may be sometimes settled by observing that the greater side of a triangle is opposite to the greater angle. Or we may determine c from equation (1) of Art. 54, which is free from ambiguity.

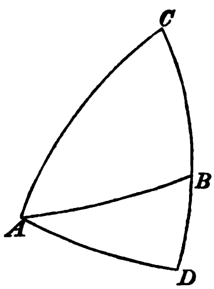
Or we may determine c, without previously determining A and B, from the formula $\cos c = \cos a \cos b + \sin a \sin b \cos C$; this is free from ambiguity. This formula may be adapted to logarithms thus;

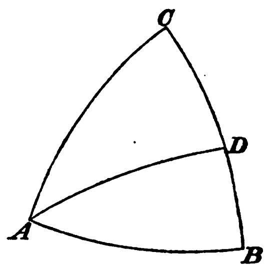
$$\cos c = \cos b (\cos a + \sin a \tan b \cos C);$$

assume $\tan\theta = \tan b \cos C$; then

$$\cos c = \cos b\,(\cos a + \sin a \tan\theta) = \frac{\cos b \cos(a-\theta)}{\cos\theta};$$

this is adapted to logarithms.





Or we may treat this case conveniently by resolving the triangle into the sum or difference of two right-angled triangles. From A draw the arc AD perpendicular to CB or CB produced; then, by Art. 62, $\tan CD = \tan b \cos C$, and this determines CD, and then DB is known. Again, by Art. 62,

$$\cos c = \cos AD \cos DB = \cos DB \frac{\cos b}{\cos CD};$$

this finds c. It is obvious that CD is what was denoted by θ in the former part of the Article.

By Art. 62,

$$\tan AD = \tan C \sin CD, \text{ and } \tan AD = \tan ABD \sin DB;$$

$$\text{thus } \tan ABD \sin DB = \tan C \sin\theta,$$

where $DB = a - \theta$ or $\theta - a$, according as D is on CB or CB produced, and ABD is either B or the supplement of B; this formula enables us to find B independently of A.

Thus, in the present case, there is no real ambiguity, and the triangle is always possible.

83. *Having given two angles and the included side* (A, c, B).

By Napier's analogies,

$$\tan \tfrac{1}{2}(a+b) = \frac{\cos \frac{1}{2}(A-B)}{\cos \frac{1}{2}(A+B)} \tan \tfrac{1}{2} c,$$

$$\tan \tfrac{1}{2}(a-b) = \frac{\sin \frac{1}{2}(A-B)}{\sin \frac{1}{2}(A+B)} \tan \tfrac{1}{2} c;$$

these determine $\frac{1}{2}(a+b)$ and $\frac{1}{2}(a-b)$, and thence a and b.

Then C may be found from the formula $\sin C = \frac{\sin A \sin c}{\sin a}$; in this case, since C is found from its sine, it may be uncertain which of two values is to be given to it; the point may be sometimes settled by observing that the greater angle of a triangle is opposite to the greater side. Or we may determine C from equation (3) of Art. 54, which is free from ambiguity.

Or we may determine C without previously determining a and b from the formula $\cos C = -\cos A \cos B + \sin A \sin B \cos c$. This formula may be adapted to logarithms, thus;

$$\cos C = \cos B\,(-\cos A + \sin A \tan B \cos c);$$

assume $\cot \phi = \tan B \cos c$; then

$$\cos C = \cos B\,(-\cos A + \cot \phi \sin A) = \frac{\cos B \sin (A-\phi)}{\sin \phi};$$

this is adapted to logarithms.

Or we may treat this case conveniently by resolving the triangle into the sum or difference of two right-angled triangles. From A draw the arc AD perpendicular to CB (see the right-hand figure of Art. 82); then, by Art. 62, $\cos c = \cot B \cot DAB$, and this determines DAB, and then CAD is known. Again, by Art. 62,

$$\cos AD \sin CAD = \cos C \text{ and } \cos AD \sin BAD = \cos B;$$

therefore $$\frac{\cos C}{\sin CAD} = \frac{\cos B}{\sin BAD}; \text{ this finds } C.$$

It is obvious that DAB is what was denoted by ϕ in the former part of the Article.

By Art. 62,

$$\tan AD = \tan AC \cos CAD, \text{ and } \tan AD = \tan AB \cos BAD;$$

thus $$\tan b \cos CAD = \tan c \cos \phi,$$

where $CAD = A - \phi$; this formula enables us to find b independently of a.

Similarly we may proceed when the perpendicular AD falls on CB *produced;* (see the left-hand figure of Art. 82).

Thus, in the present case, there is no real ambiguity; moreover the triangle is always possible.

84. *Having given two sides and the angle opposite one of them* (a, b, A).

The angle B may be found from the formula

$$\sin B = \frac{\sin b}{\sin a} \sin A,$$

and then C and c from Napier's analogies,

$$\tan \tfrac{1}{2} C = \frac{\cos \frac{1}{2}(a-b)}{\cos \frac{1}{2}(a+b)} \cot \tfrac{1}{2}(A+B),$$

$$\tan \tfrac{1}{2} c = \frac{\cos \frac{1}{2}(A+B)}{\cos \frac{1}{2}(A-B)} \tan \tfrac{1}{2}(a+b).$$

In this case, since B is found from its sine, there will sometimes be two solutions; and sometimes there will be no solution at all, namely, when the value found for $\sin B$ is greater than unity. We will presently return to this point. (See Art. 86.)

We may also determine C and c independently of B by formulæ adapted to logarithms. For, by Art. 44,

$$\cot a \sin b = \cos b \cos C + \sin C \cot A = \cos b \left(\cos C + \frac{\cot A}{\cos b} \sin C\right);$$

assume $\tan \phi = \dfrac{\cot A}{\cos b}$; thus

$$\cot a \sin b = \cos b \, (\cos C + \tan \phi \sin C) = \frac{\cos b \cos (C - \phi)}{\cos \phi};$$

therefore $$\cos (C - \phi) = \cos \phi \cot a \tan b;$$

from this equation $C-\phi$ is to be
guity still exists; for if the last
will be satisfied also by $\phi - C = a$
values for C, if $\phi + a$ is less than π

And

$\cos a = \cos b \cos c + \sin b \sin c \cos A$

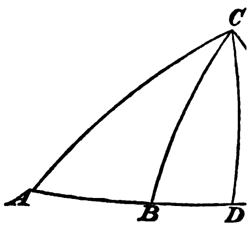
assume $\tan \theta = \tan b \cos A$; thus

$$\cos a = \cos b \; (\cos c + \sin c$$

therefore $\cos (c-\theta) =$

from this equation $c-\theta$ is to be fc
be an ambiguity as before.

Or we may treat this case cc
angle into the sum or difference of



Let $CA = b$, and let $CAE =$ th
CD perpendicular to AE, and let
shews that there may be two triar
ments. Then, by Art. 62, $\cos b =$
Again, by Art. 62,

$\tan CD = \tan AC \cos ACD$,

and $\tan CD = \tan CB \cos BCD$,

therefore $\tan AC \cos ACD = \tan C$
this finds BCD or $B'CD$.

It is obvious that ACD is what
part of the Article.

Also, by Art. 62, $\tan AD = \tan AC \cos A$; this finds AD. Then

$$\cos AC = \cos CD \cos AD, \quad \cos CB = \cos CD \cos BD,$$

$$\text{or} \quad \cos CB' = \cos CD \cos B'D;$$

therefore $$\frac{\cos AC}{\cos AD} = \frac{\cos CB}{\cos BD} \text{ or } \frac{\cos CB'}{\cos B'D};$$

this finds BD or $B'D$.

It is obvious that AD is what was denoted by θ in the former part of the Article.

85. *Having given two angles and the side opposite one of them* (A, B, a).

This case is analogous to that immediately preceding, and gives rise to the same ambiguities. The side b may be found from the formula $\sin b = \dfrac{\sin B \sin a}{\sin A}$, and then C and c from Napier's analogies,

$$\tan \tfrac{1}{2} C = \frac{\cos \frac{1}{2}(a-b)}{\cos \frac{1}{2}(a+b)} \cot \tfrac{1}{2}(A+B),$$

$$\tan \tfrac{1}{2} c = \frac{\cos \frac{1}{2}(A+B)}{\cos \frac{1}{2}(A-B)} \tan \tfrac{1}{2}(a+b).$$

We may also determine C and c independently of b by formulæ adapted to logarithms. For

$$\cos A = -\cos B \cos C + \sin B \sin C \cos a$$
$$= \cos B(-\cos C + \tan B \sin C \cos a),$$

assume $\cot \phi = \tan B \cos a$; thus

$$\cos A = \cos B(-\cos C + \sin C \cot \phi) = \frac{\cos B \sin (C-\phi)}{\sin \phi};$$

therefore $$\sin (C-\phi) = \frac{\cos A \sin \phi}{\cos B};$$

from this equation $C-\phi$ is to be found and then C. Since $C-\phi$ is found from its sine there may be an ambiguity. Again, by Art. 44,

$\cot A \sin B = \cot a \sin c - \cos c \cos B =$

assume $\cot \theta = \dfrac{\cot a}{\cos B}$; then

$$\cot A \sin B = \cos B (-\cos c + \sin c$$

therefore $\sin (c - \theta) = \cot A$ t

from this equation $c - \theta$ is to be founc
found from its sine there may be an a
be shewn that these results agree witl
the triangle into two right-angled tri
ACB' the arc CD be drawn perpe
will $= \phi$, and $B'D = \theta$.

86. We now return to the con
which may occur in the case of Art.
and the angle opposite one of them.
tedious from its length, but presents n

Before considering the problem
particular case in which $a = b$; then
third of Napier's analogies give

$$\cot \tfrac{1}{2} C = \tan A \cos a, \qquad \text{ta}$$

now $\cot \frac{1}{2} C$ and $\tan \frac{1}{2} c$ must both be
be of the same affection. Hence, whe
tion at all, unless A and a are of t
there will be only one solution; exc
right angles, and then $\cot \frac{1}{2} C$ and ta
there is an infinite number of solution

We now proceed to the general di

If $\sin b \sin A$ be greater than sin
satisfies the given conditions; if sin
$\sin a$, the equation $\sin B = \dfrac{\sin b \sin A}{\sin a}$
which we will denote by β and β', so
pose that β is the one which is not gr

Now, in order that these values of B may be admissible, it is necessary and sufficient that the values of $\cot\frac{1}{2}C$ and of $\tan\frac{1}{2}c$ should both be positive, that is, $A-B$ and $a-b$ must have the same sign by the second and fourth of Napier's analogies. We have therefore to compare the sign of $A-\beta$ and the sign of $A-\beta'$ with that of $a-b$. We will suppose that A is less than a right angle, and separate the corresponding discussion into three cases.

I. Let b be less than $\frac{\pi}{2}$.

(1) Let a be less than b; the formula $\sin B = \frac{\sin b}{\sin a}\sin A$ makes β greater than A, and *à fortiori* β' greater than A. Hence there are two solutions.

(2) Let a be equal to b; then there is one solution, as previously shewn.

(3) Let a be greater than b; we may have then $a+b$ less than π or equal to π or greater than π. If $a+b$ is less than π, then $\sin a$ is greater than $\sin b$; thus β is less than A and therefore admissible, and β' is greater than A and inadmissible. Hence there is one solution. If $a+b$ is equal to π, then β is equal to A, and β' greater than A, and both are inadmissible. Hence there is no solution. If $a+b$ is greater than π, then $\sin a$ is less than $\sin b$, and β and β' are both greater than A, and both inadmissible. Hence there is no solution.

II. Let b be equal to $\frac{\pi}{2}$.

(1) Let a be less than b; then β and β' are both greater than A, and both admissible. Hence there are two solutions.

(2) Let a be equal to b; then there is no solution, as previously shewn.

(3) Let a be greater than b; then $\sin a$ is less than $\sin b$, and β and β' are both greater than A, and inadmissible. Hence there is no solution.

III. Let b be greater than $\frac{\pi}{2}$.

(1) Let a be less than b; we may have then $a+b$ less than π or equal to π or greater than π. If $a+b$ is less than π, then $\sin a$ is less than $\sin b$, and β and β' are both greater than A and both admissible. Hence there are two solutions. If $a+b$ is equal to π, then β is equal to A and is inadmissible, and β' is greater than A and admissible. Hence there is one solution. If $a+b$ is greater than π then $\sin a$ is greater than $\sin b$; β is less than A and inadmissible, and β' greater than A and admissible. Hence there is one solution.

(2) Let a be equal to b; then there is no solution, as previously shewn.

(3) Let a be greater than b; then $\sin a$ is less than $\sin b$, and β and β' are both greater than A and both inadmissible. Hence there is no solution.

We have then the following results when A *is less than a right angle:*

$$b < \frac{\pi}{2} \quad \begin{cases} a < b \text{ .... two solutions,} \\ a = b \text{ .... one solution,} \\ a > b \text{ and } a+b < \pi \text{ .... one solution,} \\ a > b \text{ and } a+b = \pi \text{ or } > \pi \text{ ...... no solution.} \end{cases}$$

$$b = \frac{\pi}{2} \quad \begin{cases} a < b \text{ .... two solutions,} \\ a = b \text{ or } a > b \text{ .... no solution.} \end{cases}$$

$$b > \frac{\pi}{2} \quad \begin{cases} a < b \text{ and } a+b < \pi \text{ .... two solutions,} \\ a < b \text{ and } a+b = \pi \text{ or } > \pi \text{ ...... one solution,} \\ a = b \text{ or } > b \text{ .... no solution.} \end{cases}$$

It must be remembered, however, that in the cases in which two solutions are indicated, there will be no solution at all if $\sin a$ be less than $\sin b \sin A$.

In the same manner the cases in which A is equal to a right angle or greater than a right angle may be discussed, and the following results obtained:

When A is equal to a right angle,

$$b < \frac{\pi}{2} \quad \left\{ \begin{array}{ll} a < b \text{ or } a = b \quad \ldots\ldots & \text{no solution,} \\ a > b \text{ and } a + b < \pi \quad \ldots\ldots & \text{one solution,} \\ a > b \text{ and } a + b = \pi \text{ or } > \pi \quad \ldots\ldots & \text{no solution.} \end{array} \right.$$

$$b = \frac{\pi}{2} \quad \left\{ \begin{array}{ll} a < b \text{ or } a > b \quad \ldots\ldots & \text{no solution,} \\ a = b \quad \ldots\ldots & \text{infinite number of solutions.} \end{array} \right.$$

$$b > \frac{\pi}{2} \quad \left\{ \begin{array}{ll} a < b \text{ and } a + b > \pi \quad \ldots\ldots & \text{one solution,} \\ a < b \text{ and } a + b = \pi \text{ or } < \pi \quad \ldots\ldots & \text{no solution,} \\ a = b \text{ or } a > b \quad \ldots\ldots & \text{no solution.} \end{array} \right.$$

When A is greater than a right angle,

$$b < \frac{\pi}{2} \quad \left\{ \begin{array}{ll} a < b \text{ or } a = b \quad \ldots\ldots & \text{no solution,} \\ a > b \text{ and } a + b = \pi \text{ or } < \pi \quad \ldots\ldots & \text{one solution,} \\ a > b \text{ and } a + b > \pi \quad \ldots\ldots & \text{two solutions.} \end{array} \right.$$

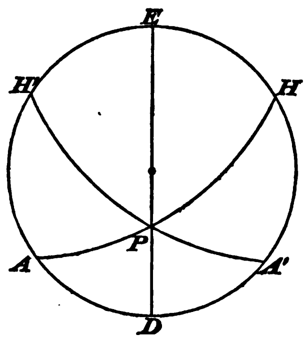
$$b = \frac{\pi}{2} \quad \left\{ \begin{array}{ll} a < b \text{ or } a = b \quad \ldots\ldots & \text{no solution,} \\ a > b \quad \ldots\ldots & \text{two solutions.} \end{array} \right.$$

$$b > \frac{\pi}{2} \quad \left\{ \begin{array}{ll} a < b \text{ and } a + b > \pi \quad \ldots\ldots & \text{one solution,} \\ a < b \text{ and } a + b = \pi \text{ or } < \pi \quad \ldots\ldots & \text{no solution,} \\ a = b \quad \ldots\ldots & \text{one solution,} \\ a > b \quad \ldots\ldots & \text{two solutions.} \end{array} \right.$$

As before in the cases in which two solutions are indicated, there will be no solution at all if $\sin a$ be less than $\sin b \sin A$.

It will be seen from the above investigations that if a lies between b and $\pi - b$, there will be one solution; if a does not lie between b and $\pi - b$ either there are two solutions or there is no solution; this enunciation is not meant to include the cases in which $a = b$ or $= \pi - b$.

87. The results of the preceding article may be illustrated by a figure.



Let $ADA'E$ be a great circle; suppose PA and PA' the projections on the plane of this circle of arcs which are each equal to b and inclined at an angle A to ADA'; let PD and PE be the projections of the least and greatest distances of P from the great circle, (see Art. 59). Thus the figure supposes A and b each less than $\frac{\pi}{2}$.

If a be less than the arc which is represented by PD there is no triangle; if a be between PD and PA in magnitude, there are two triangles, since B will fall on ADA', and we have two triangles BPA and BPA'; if a be between PA and PH there will be only one triangle, as B will fall on $A'H$ or AH', and the triangle will be either APB with B between A' and H, or else $A'PB$ with B between A and H'; but these two triangles are symmetrically equal (Art. 57); if a be greater than PH there will be no triangle. The figure will easily serve for all the cases; thus if A is greater than $\frac{\pi}{2}$, we can suppose PAE and $PA'E$ to be equal to A; if b is greater than $\frac{\pi}{2}$, we can take PH and PH' to represent b.

88. The ambiguities which occur in the last case in the so tion of oblique-angled triangles (Art. 85) may be discussed in t same manner as those in Art. 86; or by means of the po triangle, the last case may be deduced from that of Art. 86.

EXAMPLES.

1. The sides of a triangle are 105°, 90°, and 75° respective find the sines of all the angles.

2. Prove that $\tan \frac{1}{2} A \tan \frac{1}{2} B = \frac{\sin (s-c)}{\sin s}$. Solve a trian when a side, an adjacent angle, and the sum of the other t sides are given.

3. Solve a triangle having given a side, an adjacent ang and the sum of the other two angles.

4. A triangle has the sum of two sides equal to a semic cumference; find the arc joining the vertex with the mid of the base.

5. If a, b, c are known, c being a *quadrant*, determine t angles; shew also that if δ be the perpendicular on c from t opposite angle, $\cos^2 \delta = \cos^2 a + \cos^2 b$.

6. If one side of a spherical triangle be divided into fo equal parts, and θ_1, θ_2, θ_3, θ_4 be the angles subtended at the op site angle by the parts taken in order, shew that

$$\sin (\theta_1 + \theta_2) \sin \theta_2 \sin \theta_4 = \sin (\theta_3 + \theta_4) \sin \theta_1 \sin \theta_3.$$

7. In a spherical triangle if $A = B = 2C$ shew that

$$8 \sin \left(a + \frac{c}{2}\right) \sin^2 \frac{c}{2} \cos \frac{c}{2} = \sin^3 a.$$

8. In a spherical trian;

$$8 \sin^2 \frac{C}{2} \Big(\cos$$

9. If the equal sides o
by an arc DE, and BC be t

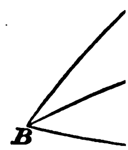
$$\sin \frac{DE}{2} =$$

10. If c_1, c_2 be the tw
are given and the triangle i

$$\tan \frac{c_1}{2} \tan \frac{c_2}{2} =$$

VII. CIRCUMSCRIBE

89. *To find the angul*
in a given triangle.



Let ABC be the triangl
meeting in P; from P dra
sides. Then it may be she
also that $AE = AF$, $BF = B$
the sum of the sides $= s$; th

Now $\tan PF =$ ta

thus $\tan r =$ tan

The value of tan r may be expressed in various forms; thus from Art. 45, we obtain

$$\tan\frac{A}{2}=\sqrt{\left\{\frac{\sin(s-b)\sin(s-c)}{\sin s\sin(s-a)}\right\}};$$

substitute this value in (1), thus

$$\tan r=\sqrt{\left\{\frac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s}\right\}}=\frac{n}{\sin s}\text{ (Art. 46)}\ldots(2).$$

Again

$$\sin(s-a)=\sin\{\tfrac{1}{2}(b+c)-\tfrac{1}{2}a\}$$

$$=\sin\tfrac{1}{2}(b+c)\cos\tfrac{1}{2}a-\cos\tfrac{1}{2}(b+c)\sin\tfrac{1}{2}a$$

$$=\frac{\sin\frac{1}{2}a\cos\frac{1}{2}a}{\sin\frac{1}{2}A}\{\cos\tfrac{1}{2}(B-C)-\cos\tfrac{1}{2}(B+C)\},\text{ (Art. 54)}$$

$$=\frac{\sin a\sin\frac{1}{2}B\sin\frac{1}{2}C}{\sin\frac{1}{2}A};$$

therefore from (1) $\tan r=\dfrac{\sin\frac{1}{2}B\sin\frac{1}{2}C}{\cos\frac{1}{2}A}\sin a$ ....(3);

hence by Art. 51,

$$\tan r=\frac{\sqrt{\{-\cos S\cos(S-A)\cos(S-B)\cos(S-C)\}}}{2\cos\frac{1}{2}A\cos\frac{1}{2}B\cos\frac{1}{2}C}$$

$$=\frac{N}{2\cos\frac{1}{2}A\cos\frac{1}{2}B\cos\frac{1}{2}C}\ldots\ldots\ldots\ldots(4).$$

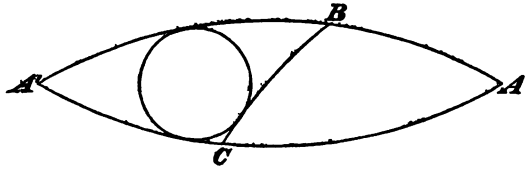
It may be shewn by common trigonometrical formulæ that

$$4\cos\tfrac{1}{2}A\cos\tfrac{1}{2}B\cos\tfrac{1}{2}C=\cos S+\cos(S-A)+\cos(S-B)+\cos(S-C);$$

hence we have from (4)

$$\cot r=\frac{1}{2N}\left\{\cos S+\cos(S-A)+\cos(S-B)+\cos(S-C)\right\}\ldots\ldots(5).$$

90. *To find the angular radius of the small circle described so as to touch one side of a given triangle, and the other sides produced.*



Let ABC be the triangle; and suppose we require the radius of the small circle which touches BC, and AB and AC produced. Produce AB and AC to meet in A'; then we require the radius of the small circle *inscribed in* A'BC, and the sides of $A'BC$ are a, $\pi - b$, $\pi - c$ respectively. Hence if r_1 be the required radius, and s denote as usual $\frac{1}{2}(a+b+c)$, we have from Art. 89,

$$\tan r_1 = \tan \frac{A}{2} \sin s \text{ .... (1).}$$

From this result we may derive other equivalent forms as in the preceding article; or we may make use of those forms immediately, observing that the angles of the triangle $A'BC$ are A, $\pi - B$, $\pi - C$ respectively. Hence s being $\frac{1}{2}(a+b+c)$ and S being $\frac{1}{2}(A+B+C)$ we shall obtain

$$\tan r_1 = \sqrt{\left\{\frac{\sin s \sin (s-b) \sin (s-c)}{\sin (s-a)}\right\}} = \frac{n}{\sin (s-a)} \cdots (2),$$

$$\tan r_1 = \frac{\cos \frac{1}{2} B \cos \frac{1}{2} C}{\cos \frac{1}{2} A} \sin a \text{ .... (3),}$$

$$\tan r_1 = \frac{\sqrt{\{-\cos S \cos (S-A) \cos (S-B) \cos (S-C)\}}}{2 \cos \frac{1}{2} A \sin \frac{1}{2} B \sin \frac{1}{2} C}$$

$$= \frac{N}{2 \cos \frac{1}{2} A \sin \frac{1}{2} B \sin \frac{1}{2} C} \text{ .... (4),}$$

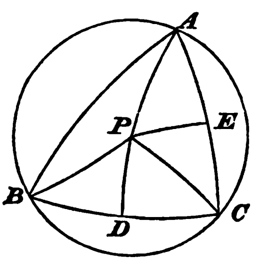
$$\cot r_1 = \frac{1}{2N}\left\{-\cos S - \cos (S-A) + \cos(S-B) + \cos(S-C)\right\} \ldots (5).$$

These results may also be found independently by bisecting two of the angles of the triangle $A'BC$, so as to determine the pole of the small circle, and proceeding as in Art. 89.

91. A circle which touches one side of a triangle and the other sides produced is called an *escribed circle;* thus there are three escribed circles belonging to a given triangle. We may denote the radii of the escribed circles which touch CA and AB respectively by r_2 and r_3, and values of tan r_2 and tan r_3 may be found from what has been already given with respect to tan r_1 by appropriate changes in the letters which denote the sides and angles.

In the preceding article a triangle $A'BC$ was formed by producing AB and AC to meet again in A'; similarly another triangle may be formed by producing BC and BA to meet again, and another by producing CA and CB to meet again. The original triangle ABC and the three formed from it have been called *associated triangles*, ABC being the fundamental triangle. Thus the inscribed and escribed circles of a given triangle are the same as the circles inscribed in the system of associated triangles of which the given triangle is the fundamental triangle.

92. *To find the angular radius of the small circle described about a given triangle.*



Let ABC be the given triangle; bisect the sides CB, CA in D and E respectively, and draw from D and E arcs perpendicular to CB and CA respectively, and let P be the intersection of these

arcs. Then P will be the po
ABC. For draw PA, PB,
triangles PCD and PBD it
the right-angled triangles PC
hence $PA = PB = PC$. Also
the angle $PBC =$ the angle l
PAC; therefore $PCB + A =$

Let $PC = R$.

Now $\tan CD = \tan$

thus $\tan \frac{1}{2} a = $ ta

therefore $\tan R = \frac{\text{ta}}{\cos (}$

The value of $\tan R$ may b

if we substitute for $\tan \frac{a}{2}$ from

$$\tan R = \sqrt{\left\{\frac{- \text{c}}{\cos (S - A) \cos (S}\right.}$$

Again $\cos (S - A) = \cos \{\frac{1}{2} (B$

$= \cos \frac{1}{2} (B$

$= \frac{\sin \frac{1}{2} A \cos \frac{1}{2} A}{\cos \frac{1}{2} a}$

$= \frac{\sin A}{\cos \frac{1}{2} a} \cos \frac{1}{2} b$

therefore from (1)

$$\tan R = \frac{\sin \frac{1}{2} a}{\sin A \cos \frac{1}{2} b \text{ co}}$$

Substitute in the last ex
Art. 46; thus

$$\tan R = \frac{2 \sin \frac{1}{2} a \sin \frac{1}{2}}{\sqrt{\{\sin s \sin (s - a) \sin (}}$$

$= \frac{2 \text{ s}}{}$

It may be shewn by common trigonometrical formulæ, that

$$4 \sin \tfrac{1}{2} a \sin \tfrac{1}{2} b \sin \tfrac{1}{2} c = \sin (s-a) + \sin (s-b) + \sin (s-c) - \sin s;$$

hence we have from (4)

$$\tan R = \frac{1}{2n} \left\{ \sin (s-a) + \sin (s-b) + \sin (s-c) - \sin s \right\} \quad\ldots\ldots\ldots(5).$$

93. *To find the angular radii of the small circles described round the triangles associated with a given fundamental triangle.*

Let R_1 denote the radius of the circle described round the triangle formed by producing AB and AC to meet again in A'; similarly let R_2 and R_3 denote the radii of the circles described round the other two triangles which are similarly formed. Then we may deduce expressions for $\tan R_1$, $\tan R_2$, and $\tan R_3$ from those found in Art. 92 for $\tan R$. The sides of the triangle $A'BC$ are a, $\pi - b$, $\pi - c$, and its angles are A, $\pi - B$, $\pi - C$; hence if $s = \frac{1}{2}(a+b+c)$ and $S = \frac{1}{2}(A+B+C)$ we shall obtain from Art. 92

$$\tan R_1 = \frac{\tan \frac{1}{2} a}{-\cos S} \quad\ldots\ldots\ldots(1),$$

$$\tan R_1 = \sqrt{\left\{ \frac{\cos (S-A)}{-\cos S \cos (S-B) \cos (S-C)} \right\}} = \frac{\cos (S-A)}{N} \quad\ldots(2),$$

$$\tan R_1 = \frac{\sin \frac{1}{2} a}{\sin A \sin \frac{1}{2} b \sin \frac{1}{2} c} \quad\ldots\ldots\ldots(3),$$

$$\tan R_1 = \frac{2 \sin \frac{1}{2} a \cos \frac{1}{2} b \cos \frac{1}{2} c}{\sqrt{\{\sin s \sin (s-a) \sin (s-b) \sin (s-c)\}}} \quad\ldots\ldots\ldots(4),$$

$$\tan R_1 = \frac{1}{2n} \left\{ \sin s - \sin (s-a) + \sin (s-b) + \sin (s-c) \right\} \quad\ldots\ldots\ldots(5).$$

Similarly we may find expressions for $\tan R_2$ and $\tan R_3$.

94. Many examples may be proposed involving properties of the circles inscribed in and described about the associated triangles. We will give one that will be of use hereafter.

To prove that

$$(\cot r + \tan R)^2 = \frac{1}{4n^2}(\sin a + \sin b + \sin c)^2 - 1.$$

We have

$$4n^2 = 1 - \cos^2 a - \cos^2 b - \cos^2 c + 2\cos a \cos b \cos c;$$

therefore

$$(\sin a + \sin b + \sin c)^2 - 4n^2$$

$$= 2\left(1 + \sin a \sin b + \sin b \sin c + \sin c \sin a - \cos a \cos b \cos c\right).$$

Also $\cot r + \tan R = \dfrac{1}{2n}\left\{\sin s + \sin(s-a) + \sin(s-b) + \sin(s-c)\right\}$;

and by squaring both members of this equation the required result will be obtained. For it may be shewn by reduction that

$$\sin^2 s + \sin^2(s-a) + \sin^2(s-b) + \sin^2(s-c) = 2 - 2\cos a \cos b \cos c,$$

and

$$\sin s \sin(s-a) + \sin s \sin(s-b) + \sin s \sin(s-c)$$
$$+ \sin(s-a)\sin(s-b) + \sin(s-b)\sin(s-c) + \sin(s-c)\sin(s-a)$$
$$= \sin a \sin b + \sin b \sin c + \sin c \sin a.$$

Similarly we may prove that

$$(\cot r_1 - \tan R)^2 = \frac{1}{4n^2}(\sin b + \sin c - \sin a)^2 - 1.$$

95. In the figure to Art. 89, suppose DP produced through P to a point A' such that DA' is a quadrant, then A' is a pole of BC and $PA' = \frac{\pi}{2} - r$; similarly, suppose EP produced through P to a point B' such that EB' is a quadrant, and FP produced through P to a point C' such that FC' is a quadrant. Then $A'B'C'$ is the polar triangle of ABC, and $PA' = PB' = PC' = \frac{\pi}{2} - r$. Thus P is the pole of the small circle *described round* the polar triangle, and the angular radius of the small circle described round the polar triangle is the complement of the angular radius of the

small circle inscribed in the primitive triangle. And in like manner the point which is the pole of the small circle inscribed in the polar triangle is also the pole of the small circle described round the primitive triangle, and the angular radii of the two circles are complementary.

EXAMPLES.

In the following examples the notation of the chapter is retained.

Shew that in any triangle the following relations hold contained in Examples 1 to 5:

1. $\operatorname{Tan} r_1 \tan r_2 \tan r_3 = \tan r \sin^2 s$.

2. $\operatorname{Tan} R + \cot r = \tan R_1 + \cot r_1 = \tan R_2 + \cot r_2$
$$= \tan R_3 + \cot r_3 = \tfrac{1}{2}(\cot r + \cot r_1 + \cot r_2 + \cot r_3).$$

3. $\operatorname{Tan}^2 R + \tan^2 R_1 + \tan^2 R_2 + \tan^2 R_3$
$$= \cot^2 r + \cot^2 r_1 + \cot^2 r_2 + \cot^2 r_3.$$

4. $$\frac{\operatorname{Tan} r_1 + \tan r_2 + \tan r_3 - \tan r}{\cot r_1 + \cot r_2 + \cot r_3 - \cot r} = \tfrac{1}{2}(1 + \cos a + \cos b + \cos c).$$

5. $\operatorname{Tan} R_1 \tan R_2 \tan R_3 = \tan R \sec^2 S$.

6. Shew that in an equilateral triangle $\tan R = 2 \tan r$.

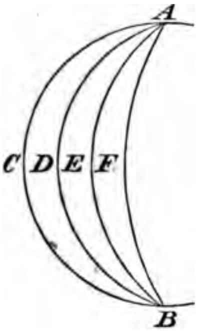
7. If ABC be an equilateral spherical triangle, P the pole of the circle circumscribing it, Q any point on the sphere, shew that
$$\cos QA + \cos QB + \cos QC = 3 \cos PA \cos PQ.$$

8. If three small circles be inscribed in a spherical triangle having each of its angles 120°, so that each touches the other two as well as two sides of the triangle, prove that the radius of each of the small circles $= 30^\circ$, and that the centres of the three small circles coincide with the angular points of the polar triangle.

VIII. AREA OF A SPHI SPHERICAL

96. *To find the area of a Lune*

A *Lune* is that portion of the su prised between two great semicircle



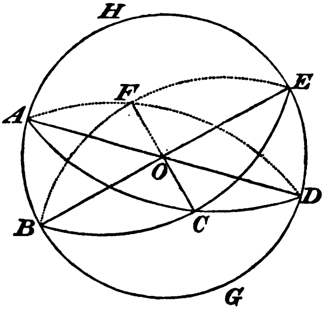
Let $ACBDA$, $ADBEA$ be two l then one of these lunes may be supp to coincide exactly with it; thus *equal.* Then by a process similar of Euclid it may be shewn that *l* *angles.* Hence since the whole su sidered as a lune with an angle e have for a lune with an angle of is A,

$$\frac{\text{area of lune}}{\text{surface of sph}}$$

Suppose r the radius of the sp (*Integral Calculus*, Chap. VII.); th

$$\text{area of lune} = \frac{A}{2\pi}\cdot$$

97. *To find the area of a Spherical Triangle.*



Let ABC be a spherical triangle; produce the arcs which form its sides until they meet again two and two, which will happen when each has become equal to the semi-circumference. The triangle ABC now forms a part of three lunes, namely, $ABDCA$, $BCEAB$, and $CAFBC$. Now the triangles CDE and AFB are subtended by vertically opposite solid angles at O, and *we will assume* that their areas are equal; therefore the lune $CAFBC$ is equal to the sum of the two triangles ABC and CDE. Hence if A, B, C denote the circular measures of the angles of the triangle, we have

$$\text{triangle } ABC + BGDC = \text{lune } ABDCA = 2Ar^2,$$

$$\text{triangle } ABC + AHEC = \text{lune } BCEAB = 2Br^2,$$

$$\text{triangle } ABC + \text{triangle } CDE = \text{lune } CAFBC = 2Cr^2;$$

hence, by addition,

twice triangle ABC + surface of hemisphere $= 2(A + B + C)\, r^2$;

therefore $\qquad$ triangle $ABC = (A + B + C - \pi)\, r^2$.

The expression $A + B + C - \pi$ is called the *spherical excess* of the triangle; and since

$$(A + B + C - \pi)\, r^2 = \frac{A + B + C - \pi}{2\pi}\, 2\pi r^2,$$

the result obtained may be thus enunciated, *the area of a spherical triangle is the same fraction of the surface of the hemisphere as the spherical excess is of four right angles.*

98. We have assumed, as is usually done, that the areas of the triangles CDE and AFB in the preceding article are equal. The triangles are, however, not absolutely equal, but *symmetrically* equal (Art. 57), so that one cannot be made to coincide with the other by superposition. It is, however, easy to decompose two such triangles into pieces which admit of superposition, and thus to prove that their areas are equal. For describe a small circle round each, then the angular radii of these circles will be equal by Art. 89. If the pole of the circumscribing circle falls inside each triangle, then each triangle is the sum of three isosceles triangles, and if the pole falls outside each triangle, then each triangle is the excess of two isosceles triangles over a third; and in each case the isosceles triangles of one set are respectively *absolutely equal* to the corresponding isosceles triangles of the other set.

99. *To find the area of a spherical polygon.*

Let n be the number of sides of the polygon, Σ the sum of all its angles. Take any point within the polygon and join it with all the angular points; thus the figure is divided into n triangles. Hence by Art. 97,

$$\text{area of polygon} = (\text{sum of the angles of the triangles} - n\pi)\, r^2,$$

and the sum of the angles of the triangles is equal to Σ together with the four right angles which are formed round the common vertex; therefore

$$\text{area of polygon} = \left\{\Sigma - (n-2)\,\pi\right\} r^2.$$

This expression is true even when the polygon has some of its angles greater than two right angles, provided it can be decomposed into triangles, of which each of the angles is less than two right angles.

100. We shall now give some expressions for certain trigonometrical functions of the *spherical excess* of a triangle. We denote the spherical excess by E, so that $E = A + B + C - \pi$.

101. *Cagnoli's Theorem.* To prove that

$$\sin \tfrac{1}{2} E = \frac{\sqrt{\{\sin s \sin (s-a) \sin (s-b) \sin (s-c)\}}}{2 \cos \tfrac{1}{2} a \cos \tfrac{1}{2} b \cos \tfrac{1}{2} c}.$$

$$\text{Sin } \tfrac{1}{2} E = \sin \tfrac{1}{2} (A + B + C - \pi) = \sin \{\tfrac{1}{2} (A + B) - \tfrac{1}{2} (\pi - C)\}$$

$$= \sin \tfrac{1}{2} (A + B) \sin \tfrac{1}{2} C - \cos \tfrac{1}{2} (A + B) \cos \tfrac{1}{2} C$$

$$= \frac{\sin \tfrac{1}{2} C \cos \tfrac{1}{2} C}{\cos \tfrac{1}{2} c} . \{\cos \tfrac{1}{2} (a - b) - \cos \tfrac{1}{2} (a + b)\}, \text{ (Art. 54)},$$

$$= \frac{\sin C \sin \tfrac{1}{2} a \sin \tfrac{1}{2} b}{\cos \tfrac{1}{2} c}$$

$$= \frac{\sin \tfrac{1}{2} a \sin \tfrac{1}{2} b}{\cos \tfrac{1}{2} c} . \frac{2}{\sin a \sin b} . \sqrt{\{\sin s \sin (s-a) \sin (s-b) \sin (s-c)\}}$$

$$= \frac{\sqrt{\{\sin s \sin (s-a) \sin (s-b) \sin (s-c)\}}}{2 \cos \tfrac{1}{2} a \cos \tfrac{1}{2} b \cos \tfrac{1}{2} c}.$$

102. *Llhuillier's Theorem.* To prove that

$$\tan \tfrac{1}{4} E = \sqrt{\{\tan \tfrac{1}{2} s \tan \tfrac{1}{2} (s-a) \tan \tfrac{1}{2} (s-b) \tan \tfrac{1}{2} (s-c)\}}.$$

$$\text{Tan } \tfrac{1}{4} E = \frac{\sin \tfrac{1}{4} (A + B + C - \pi)}{\cos \tfrac{1}{4} (A + B + C - \pi)} = \frac{\sin \tfrac{1}{2} (A + B) - \sin \tfrac{1}{2} (\pi - C)}{\cos \tfrac{1}{2} (A + B) + \cos \tfrac{1}{2} (\pi - C)},$$

(*Plane Trig.* Art. 83),

$$= \frac{\sin \tfrac{1}{2} (A + B) - \cos \tfrac{1}{2} C}{\cos \tfrac{1}{2} (A + B) + \sin \tfrac{1}{2} C} = \frac{\cos \tfrac{1}{2} (a - b) - \cos \tfrac{1}{2} c}{\cos \tfrac{1}{2} (a + b) + \cos \tfrac{1}{2} c} . \frac{\cos \tfrac{1}{2} C}{\sin \tfrac{1}{2} C},$$

(Art. 54),

$$= \frac{\sin \tfrac{1}{4} (c + a - b) \sin \tfrac{1}{4} (c + b - a)}{\cos \tfrac{1}{4} (a + b + c) \cos \tfrac{1}{4} (a + b - c)} \sqrt{\left\{\frac{\sin s \sin (s-c)}{\sin (s-a) \sin (s-b)}\right\}},$$

(Art. 45),

$$= \sqrt{\{\tan \tfrac{1}{2} s \tan \tfrac{1}{2} (s-a) \tan \tfrac{1}{2} (s-b) \tan \tfrac{1}{2} (s-c)\}}.$$

103. We may obtain many other formulæ involving trigonometrical functions of the spherical excess. Thus, for example,

$$\cos \tfrac{1}{2}E = \cos \{\tfrac{1}{2}(A+B) - \tfrac{1}{2}(\pi - C)\}$$
$$= \cos \tfrac{1}{2}(A+B) \sin \tfrac{1}{2} C + \sin \tfrac{1}{2}(A+B) \cos \tfrac{1}{2} C$$
$$= \left\{\cos \tfrac{1}{2}(a+b) \sin^2 \tfrac{1}{2}C + \cos \tfrac{1}{2}(a-b) \cos^2 \tfrac{1}{2}C\right\} \sec \tfrac{1}{2}c, \text{(Art. 54)},$$
$$= \left\{\cos \tfrac{1}{2} a \cos \tfrac{1}{2} b (\cos^2 \tfrac{1}{2} C + \sin^2 \tfrac{1}{2} C) + \sin \tfrac{1}{2} a \sin \tfrac{1}{2} b (\cos^2 \tfrac{1}{2}C - \sin^2 \tfrac{1}{2}C)\right\} \sec \tfrac{1}{2} c$$
$$= \{\cos \tfrac{1}{2} a \cos \tfrac{1}{2} b + \sin \tfrac{1}{2} a \sin \tfrac{1}{2} b \cos C\} \sec \tfrac{1}{2} c.$$

Again, it was shewn in Art. 101, that

$$\sin \tfrac{1}{2} E = \sin C \sin \tfrac{1}{2} a \sin \tfrac{1}{2} b \sec \tfrac{1}{2} c;$$

therefore $$\tan \tfrac{1}{2} E = \frac{\sin \tfrac{1}{2} a \sin \tfrac{1}{2} b \sin C}{\cos \tfrac{1}{2} a \cos \tfrac{1}{2} b + \sin \tfrac{1}{2} a \sin \tfrac{1}{2} b \cos C}.$$

Again, we have from above

$$\cos \tfrac{1}{2} E = \left\{\cos \tfrac{1}{2} a \cos \tfrac{1}{2} b + \sin \tfrac{1}{2} a \sin \tfrac{1}{2} b \cos C\right\} \sec \tfrac{1}{2} c.$$
$$= \frac{(1+\cos a)(1+\cos b) + \sin a \sin b \cos C}{4 \cos \tfrac{1}{2} a \cos \tfrac{1}{2} b \cos \tfrac{1}{2} c}$$
$$= \frac{1+\cos a + \cos b + \cos c}{4 \cos \tfrac{1}{2} a \cos \tfrac{1}{2} b \cos \tfrac{1}{2} c} = \frac{\cos^2 \tfrac{1}{2} a + \cos^2 \tfrac{1}{2} b + \cos^2 \tfrac{1}{2} c - 1}{2 \cos \tfrac{1}{2} a \cos \tfrac{1}{2} b \cos \tfrac{1}{2} c}.$$

EXAMPLES.

1. Find the angles and sides of an equilateral triangle whose area is one-fourth of that of the sphere on which it is described.

2. Find the surface of an equilateral and equiangular spherical polygon of n sides, and determine the value of each of the angles when the surface equals half the surface of the sphere.

3. If $a = b = \frac{\pi}{3}$, and $c = \frac{\pi}{2}$, shew that $E = \cos^{-1} \frac{7}{9}$.

4. If the angle C of a spherical triangle be a right an;
shew that

$$\sin \tfrac{1}{2} E = \sin \tfrac{1}{2} a \sin \tfrac{1}{2} b \sec \tfrac{1}{2} c, \quad \cos \tfrac{1}{2} E = \cos \tfrac{1}{2} a \cos \tfrac{1}{2} b \sec \tfrac{1}{2}$$

5. If the angle C be a right angle, shew that

$$\frac{\sin^2 c}{\cos c} \cos E = \frac{\sin^2 a}{\cos a} + \frac{\sin^2 b}{\cos b}.$$

6. If $a = b$ and $C = \frac{\pi}{2}$, shew that $\tan E = \frac{\sin^2 a}{2 \cos a}$.

7. The sum of the angles in a right-angled triangle is
than four right angles.

8. Draw through a given point in the side of a spher
triangle an arc of a great circle cutting off a given part of
triangle.

9. If the angles of a spherical triangle be together equal
four right angles

$$\cos^2 \tfrac{1}{2} a + \cos^2 \tfrac{1}{2} b + \cos^2 \tfrac{1}{2} c = 1.$$

10. If r_1, r_2, r_3 be the radii of three small circles o
sphere of radius r which touch one another in P, Q, R :
A, B, C be the angles of the spherical triangle formed by join
their centres,

$$\text{area } PQR = (A \cos r_1 + B \cos r_2 + C \cos r_3 - \pi)\, r^2.$$

11. Shew that

$$\sin s = \frac{\left\{\sin \tfrac{1}{2} E \sin (A - \tfrac{1}{2} E) \sin (B - \tfrac{1}{2} E) \sin (C - \tfrac{1}{2} E)\right\}^{\frac{1}{2}}}{2 \sin \tfrac{1}{2} A \sin \tfrac{1}{2} B \sin \tfrac{1}{2} C}.$$

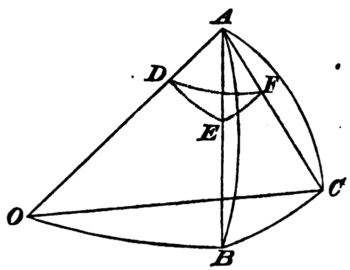
12. Given two sides of a spherical triangle, determine w
the area is a maximum.

13. Find the area of a regular polygon of a given numbe
sides formed by arcs of great circles on the surface of a sphe
and hence deduce that, if a be the angular radius of a sn
circle, its area is to that of the whole surface of the spher
versin a to 2.

IX. ON CERTAIN APPROXIMATE FORMULÆ.

104. We shall now investigate certain approximate formulæ which are often useful in calculating spherical triangles when the radius of the sphere is large compared with the lengths of the sides of the triangles.

105. *Given two sides and the included angle of a spherical triangle, to find the angle between the chords of these sides.*



Let AB, AC be the two sides of the triangle ABC; let O be the centre of the sphere. Describe a sphere round A as a centre, and suppose it to meet AO, AB, AC in D, E, F respectively. Then the angle EDF is the inclination of the planes OAB, OAC, and is therefore equal to A. From the spherical triangle DEF

$$\cos EF = \cos DE \cos DF + \sin DE \sin DF \cos A;$$

and $$DE = \tfrac{1}{2}(\pi - c), \quad DF = \tfrac{1}{2}(\pi - b);$$

therefore $$\cos EF = \sin \tfrac{1}{2} b \sin \tfrac{1}{2} c + \cos \tfrac{1}{2} b \cos \tfrac{1}{2} c \cos A.$$

If the sides of the triangle are small compared with the radius of the sphere, EF will not differ much from A; suppose $EF = A - \theta$, then approximately

$$\cos EF = \cos A + \theta \sin A;$$

and $$\sin \tfrac{1}{2} b \sin \tfrac{1}{2} c = \sin^2 \tfrac{1}{4} (b + c) - \sin^2 \tfrac{1}{4} (b - c),$$

$$\cos \tfrac{1}{2} b \cos \tfrac{1}{2} c = \cos^2 \tfrac{1}{4} (b + c) - \sin^2 \tfrac{1}{4} (b - c);$$

therefore

$$\cos A + \theta \sin A = \sin^2 \tfrac{1}{4} (b + c) - \sin^2 \tfrac{1}{4} (b - c) + \left\{1 - \sin^2 \tfrac{1}{4} (b + c) - \sin^2 \tfrac{1}{4} (b - c)\right\} \cos A;$$

therefore

$$\theta \sin A = (1 - \cos A) \sin^2 \tfrac{1}{4} (b + c) - (1 + \cos A) \sin^2 \tfrac{1}{4} (b - c),$$

therefore $$\theta = \tan \tfrac{1}{2} A \sin^2 \tfrac{1}{4} (b + c) - \cot \tfrac{1}{2} A \sin^2 \tfrac{1}{4} (b - c).$$

This gives the *circular measure* of θ; the number of seconds in the angle is found by dividing the circular measure by the circular measure of one second, or approximately by the sine of one second (*Plane Trigonometry*, Art. 123). If the lengths of the arcs corresponding to a and b respectively be α and β, and r the radius of the sphere, we have $\frac{\alpha}{r}$ and $\frac{\beta}{r}$ as the circular measures of a and b respectively; and the lengths of the sides of the chordal triangle are $2r \sin \frac{\alpha}{2r}$ and $2r \sin \frac{\beta}{2r}$ respectively. Thus when the sides of the spherical triangle and the radius of the sphere are known, we can calculate the angles and sides of the chordal triangle.

106. Legendre's Theorem. *If the sides of a spherical triangle be small compared with the radius of the sphere, then each angle of the spherical triangle exceeds by one third of the spherical excess the corresponding angle of the plane triangle, the sides of which are of the same length as the arcs of the spherical triangle.*

Let A, B, C be the angles of the spherical triangle; a, b, c the sides; r the radius of the sphere; α, β, γ the lengths of the arcs which form the sides, so that $\frac{\alpha}{r}$, $\frac{\beta}{r}$, $\frac{\gamma}{r}$ are the circular measures of a, b, c respectively. Then

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c};$$

now $$\cos a = 1 - \frac{a^2}{2r^2} + \frac{a^4}{24r^4} - \ldots,$$

$$\sin a = \frac{a}{r} - \frac{a^3}{6r^3} + \ldots$$

Similar expressions hold for $\cos b$ and $\sin b$, and for $\cos c$ and $\sin c$ respectively. Hence, if we neglect powers of the circular measure above the *fourth*, we have

$$\cos A = \frac{1 - \frac{\alpha^2}{2r^2} + \frac{\alpha^4}{24r^4} - \left(1 - \frac{\beta^2}{2r^2} + \frac{\beta^4}{24r^4}\right)\left(1 - \frac{\gamma^2}{2r^2} + \frac{\gamma^4}{24r^4}\right)}{\frac{\beta\gamma}{r^2}\left(1 - \frac{\beta^2}{6r^2}\right)\left(1 - \frac{\gamma^2}{6r^2}\right)}$$

$$= \frac{\frac{1}{2r^2}(\beta^2 + \gamma^2 - \alpha^2) + \frac{1}{24r^4}(\alpha^4 - \beta^4 - \gamma^4 - 6\beta^2\gamma^2)}{\frac{\beta\gamma}{r^2}\left(1 - \frac{\beta^2 + \gamma^2}{6r^2}\right)}$$

$$= \frac{1}{2\beta\gamma}\left\{\beta^2 + \gamma^2 - \alpha^2 + \frac{1}{12r^2}(\alpha^4 - \beta^4 - \gamma^4 - 6\beta^2\gamma^2)\right\}\left\{1 + \frac{\beta^2 + \gamma^2}{6r^2}\right\}$$

$$= \frac{\beta^2 + \gamma^2 - \alpha^2}{2\beta\gamma} + \frac{\alpha^4 + \beta^4 + \gamma^4 - 2\alpha^2\beta^2 - 2\beta^2\gamma^2 - 2\gamma^2\alpha^2}{24\beta\gamma r^2}.$$

Now let A', B', C' be the angles of the plane triangle whose sides are α, β, γ respectively; then

$$\cos A' = \frac{\beta^2 + \gamma^2 - \alpha^2}{2\beta\gamma},$$

thus $$\cos A = \cos A' - \frac{\beta\gamma \sin^2 A'}{6r^2}.$$

Suppose $A = A' + \theta$; then

$$\cos A = \cos A' - \theta \sin A' \text{ approximately};$$

therefore $$\theta = \frac{\beta\gamma \sin A'}{6r^2} = \frac{S}{3r^2},$$

where S denotes the area of the plane triangle whose sides are α, β, γ. Similarly

$$B = B' + \frac{S}{3r^2} \text{ and } C = C' + \frac{S}{3r^2};$$

hence approximately

$$A + B + C = A' + B' + C' + \frac{S}{r^2} = \pi + \frac{S}{r^2};$$

therefore $\frac{S}{r^2}$ is approximately equal to the spherical excess of the spherical triangle, and thus the theorem is established.

It will be seen that in the above approximation the area of the spherical triangle is considered equal to the area of the plane triangle which can be found with sides of the same length.

107. Legendre's Theorem may be used for the approximate solution of spherical triangles in the following manner:

(1) Suppose the three sides of a spherical triangle known; then the values of α, β, γ are known, and by the formulæ of Plane Trigonometry we can calculate S and A', B', C'; then A, B, C are known from the formulæ

$$A = A' + \frac{S}{3r^2}, \quad B = B' + \frac{S}{3r^2}, \quad C = C' + \frac{S}{3r^2}.$$

(2) Suppose two sides and the included angle of a spherical triangle known, for example A, b, c. Then

$$S = \tfrac{1}{2}\beta\gamma \sin A' = \tfrac{1}{2}\beta\gamma \sin A \text{ approximately.}$$

Then A' is known from the formula $A' = A - \frac{S}{3r^2}$. Thus in the plane triangle two sides and the included angle are known; therefore its remaining parts can be calculated, and then those of the spherical triangle become known.

(3) Suppose two sides a
in a spherical triangle know

$$\sin B' = \frac{\beta}{\alpha} \sin A'$$

and $C' = \pi - A' - B' = \pi - A -$
Hence A' is known and the
sides and the angle opposite

(4) Suppose two angle
cal triangle known, for exam

$$\text{Then } S = \frac{\gamma^2 \sin A'}{2 \sin (A'}$$

Hence in the plane triangle
known.

(5) Suppose two angles
in a spherical triangle know

$$C' = \pi - A' - B' = \pi -$$

$$S = \frac{a^2}{2}$$

which can be calculated,
known.

108. The importance o
tion of Spherical Trigonome
surface has given rise to som
to test the degree of exact
finish the present chapter wi
will serve as exercises for t
proximately the spherical e
begin with investigating a
spherical excess.

109. *To find an approximate value of the spherical excess.*

Let E denote the spherical excess; then

$$\sin \tfrac{1}{2} E = \frac{\sin \tfrac{1}{2} a \sin \tfrac{1}{2} b \sin C}{\cos \tfrac{1}{2} c};$$

therefore approximately

$$\sin \tfrac{1}{2} E = \sin C \frac{\alpha\beta}{4r^2}\left(1 - \frac{\alpha^2}{24r^2}\right)\left(1 - \frac{\beta^2}{24r^2}\right)\left(1 - \frac{\gamma^2}{8r^2}\right)^{-1}$$

$$= \sin C \frac{\alpha\beta}{4r^2}\left(1 + \frac{3\gamma^2 - \alpha^2 - \beta^2}{24r^2}\right);$$

therefore $$E = \sin C \frac{\alpha\beta}{2r^2}\left(1 + \frac{3\gamma^2 - \alpha^2 - \beta^2}{24r^2}\right) \quad \text{.................. (1)},$$

and $$\sin C = \sin\left(C' + \tfrac{1}{3} E\right) = \sin C' + \tfrac{1}{3} E \cos C'$$

$$= \sin C' + \frac{\sin C' \cos C'}{3} \frac{\alpha\beta}{2r^2} = \sin C'\left(1 + \frac{\alpha^2+\beta^2-\gamma^2}{12r^2}\right) \text{...(2)}.$$

From (1) and (2)

$$E = \sin C' \frac{\alpha\beta}{2r^2}\left(1 + \frac{\alpha^2 + \beta^2 + \gamma^2}{24r^2}\right).$$

Hence to this order of approximation the area of the spherical triangle exceeds that of the plane triangle by the fraction $\frac{\alpha^2 + \beta^2 + \gamma^2}{24r^2}$ of the latter.

110. *To find an approximate value of* $\frac{\sin A}{\sin B}$.

$$\frac{\text{Sin } A}{\text{Sin } B} = \frac{\sin a}{\sin b};$$

hence approximately $$\frac{\sin A}{\sin B} = \frac{\alpha\left(1 - \frac{\alpha^2}{6r^2} + \frac{\alpha^4}{120r^4}\right)}{\beta\left(1 - \frac{\beta^2}{6r^2} + \frac{\beta^4}{120r^4}\right)}$$

$$=\frac{\alpha}{\beta}\left(1-\frac{\alpha^2}{6r^2}+\frac{\alpha^4}{120r^4}+\frac{\beta^2}{6r^2}-\frac{\alpha^2\beta^2}{36r^4}-\frac{\beta^4}{120r^4}+\frac{\beta^4}{36r^4}\right)$$

$$=\frac{\alpha}{\beta}\left\{1+\frac{\beta^2-\alpha^2}{6r^2}+\frac{\alpha^4-\beta^4}{120r^4}+\frac{\beta^2(\beta^2-\alpha^2)}{36r^4}\right\}$$

$$=\frac{\alpha}{\beta}\left\{1+\frac{\beta^2-\alpha^2}{6r^2}\left(1+\frac{\beta^2}{6r^2}-\frac{\alpha^2+\beta^2}{20r^2}\right)\right\}$$

$$=\frac{\alpha}{\beta}\left\{1+\frac{\beta^2-\alpha^2}{6r^2}\left(1+\frac{7\beta^2-3\alpha^2}{60r^2}\right)\right\}.$$

111. *To express* $\cot B-\cot A$ *approximately.*

$$\operatorname{Cot} B-\cot A=\frac{1}{\sin B}\left(\cos B-\frac{\sin B}{\sin A}\cos A\right);$$

hence, approximately, by Art. 110,

$$\cot B-\cot A=\frac{1}{\sin B}\left(\cos B-\frac{\beta}{\alpha}\cos A-\frac{\beta}{\alpha}\,\frac{\alpha^2-\beta^2}{6r^2}\cos A\right).$$

Now we have shewn in Art. 106, that approximately

$$\cos A=\frac{\beta^2+\gamma^2-\alpha^2}{2\beta\gamma}+\frac{\alpha^4+\beta^4+\gamma^4-2\alpha^2\beta^2-2\beta^2\gamma^2-2\gamma^2\alpha^2}{24\beta\gamma r^2},$$

therefore $\cos B-\frac{\beta}{\alpha}\cos A=\frac{\alpha^2-\beta^2}{\alpha\gamma}$ approximately,

and $$\cot B-\cot A=\frac{\alpha^2-\beta^2}{\alpha\gamma\sin B}-\frac{\alpha^2-\beta^2}{\alpha\gamma\sin B}\,\frac{\beta^2+\gamma^2-\alpha^2}{12r^2}$$

$$=\frac{\alpha^2-\beta^2}{\alpha\gamma\sin B}\left(1-\frac{\beta^2+\gamma^2-\alpha^2}{12r^2}\right).$$

112. The approximations in Arts. 109 and 110 are true so far as terms involving r^4; that in Art. 111 is true so far as terms involving r^2, and it will be seen that we are thus able to carry the approximations in the following article so far as terms involving r^4.

113. *To find an approximate value of the error in the length of a side of a spherical triangle when calculated by Legendre's Theorem.*

Suppose the side β known and the side α required; let 3μ denote the spherical excess which is adopted. Then the approximate value $\frac{\beta \sin(A-\mu)}{\sin(B-\mu)}$ is taken for the side of which α is the real value. Let $x = \alpha - \frac{\beta \sin(A-\mu)}{\sin(B-\mu)}$; we have then to find x approximately. Now approximately

$$\frac{\sin(A-\mu)}{\sin(B-\mu)} = \frac{\sin A - \mu \cos A - \frac{\mu^2}{2} \sin A}{\sin B - \mu \cos B - \frac{\mu^2}{2} \sin B}$$

$$= \frac{\sin A}{\sin B}\left(1 - \mu \cot A - \frac{\mu^2}{2}\right)\left(1 - \mu \cot B - \frac{\mu^2}{2}\right)^{-1}$$

$$= \frac{\sin A}{\sin B}\left\{1 + \mu(\cot B - \cot A) + \mu^2 \cot B(\cot B - \cot A)\right\}$$

$$= \frac{\sin A}{\sin B} + \frac{\mu \sin A}{\sin B}(\cot B - \cot A)(1 + \mu \cot B).$$

Also the following formulæ are true so far as terms involving r^2,

$$\frac{\sin A}{\sin B} = \frac{\alpha}{\beta}\left(1 + \frac{\beta^2 - \alpha^2}{6r^2}\right),$$

$$\cot B - \cot A = \frac{\alpha^2 - \beta^2}{\alpha\gamma \sin B}\left(1 - \frac{\beta^2 + \gamma^2 - \alpha^2}{12r^2}\right),$$

$$1 + \mu \cot B = 1 + \frac{\alpha^2 + \gamma^2 - \beta^2}{12r^2}.$$

Hence, approximately,

$$\frac{\sin A}{\sin B}(\cot B - \cot A)(1 + \mu \cot B) = \frac{\alpha^2 - \beta^2}{\beta\gamma \sin B}.$$

Therefore $$x = a - \frac{\beta \sin A}{\sin B} - \frac{\mu (a^2 -}{\gamma \sin}$$

$$= \frac{a (\beta^2 - a^2)}{6} \left\{ \frac{6\mu}{a\gamma \sin B} - \frac{1}{r^2} + \frac{3a^2 -}{60} \right.$$

If we calculate μ from the formula μ

$$x = \frac{a (\beta^2 - a^2)(3a^2 -}{360 r^4}$$

If we calculate μ from an equatio Art. 109, we have

$$\mu = \frac{a\gamma \sin B}{6r^2} \left(1 + \frac{3\beta^2 -}{2}\right.$$

therefore $$x = \frac{a (\beta^2 - a^2)(a^2 + \beta^2 -}{720 r^4}$$

MISCELLANEOUS EXAM

1. If the sides of a spherical triang B', C', so that BB', CC' are the semi respectively, prove that the arc $B'C'$ will centre of the sphere equal to the angle and AC.

2. Deduce Legendre's Theorem fron

$$\tan^2 \frac{A}{2} = \frac{\sin \frac{1}{2}(a + b - c) \sin}{\sin \frac{1}{2}(b + c - a) \sin}$$

3. Four points A, B, C, D on the joined by arcs of great circles, and E, of the arcs AC, BD; shew that

$$\cos AB + \cos BC + \cos CD + \cos DA =$$

4. If a quadrilateral $ABCD$ be ins a sphere so that two opposite angles A

extremities of a diameter the sum of the cosines of the sides is constant.

5. ABC is a spherical triangle each of whose sides is a quadrant, P any point within the triangle; shew that

$$\cos PA \cos PB \cos PC + \cot BPC \cot CPA \cot APB = 0,$$

and $$\tan ABP \tan BCP \tan CAP = 1.$$

6. If O be the middle point of an equilateral triangle ABC, and P any point upon the surface of the sphere,

$$\tfrac{1}{4}(\tan PO \tan OA)^2 (\cos PA + \cos PB + \cos PC)^2 =$$

$$\cos^2 PA + \cos^2 PB + \cos^2 PC - \cos PA \cos PB - \cos PB \cos PC - \cos PC \cos PA.$$

7. If ABC be a triangle having each side a quadrant, O the pole of the inscribed circle, P any point on the sphere, then

$$(\cos PA + \cos PB + \cos PC)^2 = 3 \cos^2 PO.$$

8. From each of three points on the surface of a sphere arcs are drawn on the surface to three other points situated on a great circle of the sphere, and their cosines are a, b, c; a', b', c'; a'', b'', c''. Shew that

$$ab''c' + a'bc'' + a''b'c = ab'c'' + a'b''c + a''bc'.$$

9. Prove the following approximate formula (Arts. 110, 111),

$$\log \beta = \log a + \log \sin B - \log \sin A + \frac{S}{3r^2} (\cot A - \cot B).$$

10. By continuing the approximation in Art. 106 so as to include the terms involving r^4, shew that approximately

$$\cos A = \cos A' - \frac{\beta\gamma \sin^2 A'}{6r^2} + \frac{\beta\gamma (a^2 - 3\beta^2 - 3\gamma^2) \sin^2 A'}{180r^4}.$$

11. From the preceding result shew that if $A = A' + \theta$ then approximately

$$\theta = \frac{\beta\gamma \sin A'}{6r^2} \left(1 + \frac{7\beta^2 + 7\gamma^2 + a^2}{120r^2}\right).$$

X. GEODETICAL OPERATIONS.

114. One of the most important applications of Trigonometry, both Plane and Spherical, is to the determination of the figure and dimensions of the Earth itself, and of any portion of its surface. We shall give a brief outline of the subject, and for further information refer to Woodhouse's *Trigonometry*, to the article *Trigonometrical Survey* in the *Penny Cyclopædia*, and to Airy's treatise on the *Figure of the Earth* in the *Encyclopædia Metropolitana*. For practical knowledge of the subject it will be necessary to study some of the published accounts of the great surveys which have been effected in different parts of the world, as for example, the *Account of the measurement of two sections of the Meridional arc of India*, by Lieut. Colonel Everest, 1847; or the *Account of the Observations and Calculations of the Principal Triangulations in the Ordnance Survey of Great Britain and Ireland*, 1858.

115. An important part of any survey consists in the measurement of a horizontal line, which is called a *base*. A level plain of a few miles in length is selected and a line is measured on it with every precaution to ensure accuracy. Rods of deal, and of metal, hollow tubes of glass, and steel chains, have been used in different surveys; the temperature is carefully observed during the operations, and allowance is made for the varying lengths of the rods or chains, which arise from variations in the temperature.

116. At various points of the country suitable stations are selected and signals erected; then by supposing lines to be drawn connecting the signals, the country is divided into a series of triangles. The angles of these triangles are observed, that is, the angles which any two signals subtend at a third. For example, suppose A and B to denote the extremities of the *base*, and C a

signal at a third point visible from A and B; then in the triangle ABC the angles ABC and BAC are observed, and then AC and BC can be calculated. Again, let D be a signal at a fourth point, such that it is visible from C and A; then the angles ACD and CAD are observed, and as AC is known, CD and AD can be calculated.

117. Besides the original *base* other lines are measured in various parts of the country surveyed, and their measured lengths are compared with their lengths obtained by calculation through a series of triangles from the original base. The degree of closeness with which the measured length agrees with the calculated length is a test of the accuracy of the survey. During the progress of the Ordnance Survey of Great Britain and Ireland, lines have been measured in various places; the last two are one near Lough Foyle in Ireland, which was measured in 1827 and 1828, and one on Salisbury Plain, which was measured in 1849. The line near Lough Foyle is nearly 8 miles long, and the line on Salisbury Plain is nearly 7 miles long; and the difference between the length of the line on Salisbury Plain as measured and as calculated from the Lough Foyle base is less than 5 inches (*An Account of the Observations*, &c. page 419).

118. There are various methods of effecting the calculations for determining the lengths of the sides of all the triangles in the survey. One method is to use the exact formulæ of Spherical Trigonometry. The radius of the Earth may be considered known very approximately; let this radius be denoted by r, then if a be the length of any arc the circular measure of the angle which the arc subtends at the centre of the earth is $\frac{a}{r}$. The formulæ of Spherical Trigonometry give expressions for the trigonometrical functions of $\frac{a}{r}$, so that $\frac{a}{r}$ may be found and then a. Since in practice $\frac{a}{r}$ is always very small, it becomes necessary to pay

attention to the methods of securing accuracy in calculations which involve the logarithmic trigonometrical functions of small angles (*Plane Trigonometry*, Art. 205).

Instead of the exact calculation of the triangles by Spherical Trigonometry, various methods of approximation have been proposed; only two of these methods however have been much used. One method of approximation consists in deducing from the angles of the spherical triangles the angles of the *chordal triangles*, and then computing the latter triangles by Plane Trigonometry (see Art. 105). The other method of approximation consists in the use of Legendre's Theorem (see Art. 106).

119. The three methods which we have indicated were all used by Delambre in calculating the triangles in the French survey (*Base du Système Métrique*, Tome III. page 7). In the earlier operations of the Trigonometrical survey of Great Britain and Ireland, the triangles were calculated by the chord method; but this has been for many years discontinued, and in place of it Legendre's Theorem has been universally adopted (*An Account of the Observations*, &c. page 244). The triangles in the Indian Survey are stated by Lieut. Colonel Everest to be computed on Legendre's Theorem.

120. If the three angles of a plane triangle be observed, the fact that their sum ought to be equal to two right angles affords a test of the accuracy with which the observations are made. We shall proceed to shew how a test of the accuracy of observations of the angles of a spherical triangle formed on the Earth's surface may be obtained by means of the *spherical excess.*

121. *The area of a spherical triangle formed on the Earth's surface being known in square feet, it is required to establish a rule for computing the spherical excess in seconds.*

Let n be the number of seconds in the spherical excess, s the number of square feet in the area of the triangle, r the number of feet in the radius of the Earth. Then if E be the circular measure of the spherical excess,

$$s = Er^2,$$

and $$E = \frac{n\pi}{180.60.60} = \frac{n}{206265} \text{ approximately;}$$

therefore $$s = \frac{nr^2}{206265}.$$

Now by actual measurement the mean length of a degree on the Earth's surface is found to be 365155 feet; thus

$$\frac{\pi r}{180} = 365155.$$

With the value of r obtained from this equation it is found by logarithmic calculation, that

$$\log n = \log s - 9.326774.$$

Hence n is known when s is known.

This formula is called General Roy's rule, as it was used by him in the Trigonometrical survey of Great Britain and Ireland. Mr Davies, however, claims it for Mr Dalby. (See Hutton's *Course of Mathematics*, by Davies, Vol. II. p. 47).

122. In order to apply General Roy's rule, we must know the area of the spherical triangle. Now the area is not known *exactly* unless the elements of the spherical triangle are known *exactly;* but it is found that in such cases as occur in practice an approximate value of the area is sufficient. Suppose, for example, that we use the area of the *plane triangle* considered in Legendre's Theorem, instead of the area of the *Spherical Triangle itself;* then it appears from Art. 109, that the error is approximately denoted by the fraction $\frac{\alpha^2 + \beta^2 + \gamma^2}{24r^2}$ of the former area, and this fraction is less than ·0001, if the sides do not exceed 100 miles in length. Or again, suppose we want to estimate the influence of errors in the angles on the calculation of the area; let the

circular measure of an erro

we ought to use $\frac{\alpha\beta \sin (C +}{2}$

approximately the ratio exp
observations h will not ex(
seconds, so that, if C be not
sensible.

123. The following exa
the triangles of the English s
writers. The observed ang
$42^\circ. 2'. 32''$, $67^\circ. 55'. 39''$, 7(
in the observations is require
angle A to be 27404·2 feet.
pression $\frac{a^2 \sin B \sin C}{2 \sin A}$, and
that $n = \cdot 23$. Now the sum
and as it ought to have bee
of the errors of the observa
be distributed among the o
the opinion of the observer
each of the observed angles
angles thus corrected for the

124. An investigation
form of a triangle, in whi(
angles will exercise the least
and although the reasoning
deserving of the attention
angles of a triangle observ(
required to find the form o
sides may be least affected
spherical excess of the tri
sufficient accuracy for pract
angles does not exceed two
excess, let these angles be al

each, so as to make their sum correct. Let A, B, C be the angles thus furnished by observation and altered if necessary; and let δA, δB and δC denote the respective errors of A, B and C. Then $\delta A + \delta B + \delta C = 0$, because by supposition the sum of A, B and C is correct. Considering the triangle as approximately plane, the true value of the side c is $\frac{a \sin (C + \delta C)}{\sin (A + \delta A)}$, that is, $\frac{a \sin (C + \delta C)}{\sin (A - \delta B - \delta C)}$.
Now approximately

$$\sin (C + \delta C) = \sin C + \delta C \cos C, \text{ (\textit{Plane Trig.} Chap. XII.)},$$

$$\sin (A - \delta B - \delta C) = \sin A - (\delta B + \delta C) \cos A.$$

Hence approximately

$$c = \frac{a \sin C}{\sin A} \left\{1 + \delta C \cot C\right\} \left\{1 - (\delta B + \delta C) \cot A\right\}^{-1}$$

$$= \frac{a \sin C}{\sin A} \left\{1 + \delta B \cot A + \delta C (\cot C + \cot A)\right\};$$

and $\cot C + \cot A = \frac{\sin (A + C)}{\sin A \sin C} = \frac{\sin B}{\sin A \sin C}$ approximately.

Hence the error of c is approximately

$$\frac{a \sin B}{\sin^2 A} \delta C + \frac{a \sin C \cos A}{\sin^2 A} \delta B.$$

Similarly the error of b is approximately

$$\frac{a \sin C}{\sin^2 A} \delta B + \frac{a \sin B \cos A}{\sin^2 A} \delta C.$$

Now it is impossible to assign exactly the signs and magnitudes of the errors δB and δC, so that the reasoning must be vague. It is obvious that to make the error small $\sin A$ must not be small. And as the sum of δA, δB and δC is zero, two of them must have the same sign, and the third the opposite sign; we may therefore consider that it is more probable that any two as δB and δC have *different* signs, than that they have the same sign.

If δB and δC have different signs the errors of b and c will e less when $\cos A$ is positive than when $\cos A$ is negative; therefore ought to be less than a right angle. And if δB and C are probably not very different, B and C should be nearly qual. These conditions will be satisfied by a triangle differing ot much from an equilateral triangle.

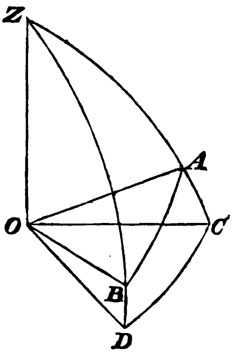
If two angles only, A and B, be observed, we obtain the same xpressions as before for the errors in b and c; but we have o reason for considering that δB and δC are of different signs ather than of the same sign. In this case then the supposition hat A is a right angle will probably make the errors smallest.

125. The preceding article is taken from the Treatise on 'rigonometry in the *Encyclopædia Metropolitana*. The least atisfactory part is that in which it is considered that δB and δC nay be supposed nearly equal; for since $\delta A + \delta B + \delta C = 0$, if we uppose δB and δC nearly equal and of opposite signs, we do in ffect suppose $\delta A = 0$ nearly; thus in observing three angles, we uppose that in one observation a certain error is made, in a econd observation the same numerical error is made but with n opposite sign, and in the remaining observation no error is nade.

126. We have hitherto proceeded on the supposition that the Earth is a sphere; it is however approximately a spheroid of small ccentricity. For the small corrections which must in consequence e introduced into the calculations we must refer to the works amed in Art. 114. One of the results obtained is that the error aused by regarding the Earth as a sphere instead of a spheroid increases with the departure of the triangle from the well-conditioned r equilateral form (*An Account of the Observations*, &c. page 243). Under certain circumstances the spherical excess is the same on a pheroid as on a sphere (*Figure of the Earth*, pages 198 and 215).

127. In geodetical operations it is sometimes required to determine the horizontal angle between two points, which are at a

small angular distance from the horizon, the angle which the objects subtend being known, and also the angles of elevation or depression.



Suppose OA and OB the directions in which the two points are seen from O; and let the angle AOB be observed. Let OZ be the direction perpendicular to the observer's horizon; describe a sphere round O as a centre, and let vertical planes through OA and OB meet the horizon in OC and OD respectively; then the angle COD is required.

Let $AOB = \theta$, $COD = \theta + x$, $AOC = h$, $BOD = k$; from the triangle AZB

$$\cos AZB = \frac{\cos\theta - \cos ZA \cos ZB}{\sin ZA \sin ZB} = \frac{\cos\theta - \sin h \sin k}{\cos h \cos k};$$

and $\cos AZB = \cos COD = \cos(\theta + x)$; thus

$$\cos(\theta + x) = \frac{\cos\theta - \sin h \sin k}{\cos h \cos k}.$$

This formula is exact; by approximation we obtain

$$\cos\theta - x \sin\theta = \frac{\cos\theta - hk}{1 - \frac{1}{2}(h^2 + k^2)};$$

therefore $$x \sin\theta = hk - \tfrac{1}{2}(h^2 + k^2)\cos\theta, \text{ nearly,}$$

and $$x = \frac{2hk - (h^2 + k^2)(\cos^2 \frac{1}{2}\theta - \sin^2 \frac{1}{2}\theta)}{2\sin\theta}$$

$$= \tfrac{1}{4}(h+k)^2 \tan\tfrac{1}{2}\theta - \tfrac{1}{4}(h-k)^2 \cot\tfrac{1}{2}\theta.$$

This process, by which we find the angle COD from the angle AOB, is called *reducing an angle to the horizon.*

XI. ON SMALL VARIATIONS IN THE PARTS OF A SPHERICAL TRIANGLE, AND ON THE CONNEXION OF FORMULÆ IN PLANE AND SPHERICAL TRIGONOMETRY.

128. It is sometimes important to know what amount of error will be introduced into one of the calculated parts of a triangle by reason of any small error which may exist in the given parts. We will here consider an example.

129. *A side and the opposite angle of a spherical triangle remain constant, determine the connexion between the small variations of any other pair of elements.*

Suppose C and c to remain constant.

(1) Required the connexion between the small variations of the other sides. We suppose a and b to denote the sides of one triangle which can be formed with C and c as fixed elements, and $a + \delta a$ and $b + \delta b$ to denote the sides of another such triangle; then we require the ratio of δa to δb when both are extremely small. We have

$$\cos c = \cos a \cos b + \sin a \sin b \cos C,$$

and $$\cos c = \cos(a + \delta a)\cos(b + \delta b) + \sin(a + \delta a)\sin(b + \delta b)\cos C;$$

also $$\cos(a + \delta a) = \cos a - \sin a\,\delta a, \text{ nearly,}$$

and $$\sin(a + \delta a) = \sin a + \cos a\,\delta a, \text{ nearly,}$$

with similar formulæ for $\cos(b+\delta b)$ and $\sin(b+\delta b)$. (See *Plane Trigonometry*, Chap. XII.). Thus

$$\cos c = (\cos a - \sin a\,\delta a)(\cos b - \sin b\,\delta b) + (\sin a + \cos a\,\delta a)(\sin b + \cos b\,\delta b)\cos C.$$

Hence by subtraction, if we neglect the product $\delta a\,\delta b$,

$$0 = \delta a\,(\sin a \cos b - \cos a \sin b \cos C) + \delta b\,(\sin b \cos a - \cos b \sin a \cos C);$$

this gives the ratio of δa to δb in terms of a, b, C. We may express the ratio more simply in terms of A and B; for, dividing by $\sin a \sin b$, we get from Art. 44,

$$\frac{\delta a}{\sin a}\cot B \sin C + \frac{\delta b}{\sin b}\cot A \sin C = 0;$$

therefore $$\delta a \cos B + \delta b \cos A = 0.$$

(2) Required the connexion between the small variations of the other angles. In this case we may by means of the polar triangle deduce from the result just found, that

$$\delta A \cos b + \delta B \cos a = 0;$$

this may also be found independently as before.

(3) Required the connexion between the small variations of a side and the opposite angle (A, a).

Here $$\sin A \sin c = \sin C \sin a,$$

and $$\sin(A + \delta A)\sin c = \sin C \sin(a + \delta a);$$

hence by subtraction

$$\cos A \sin c\,\delta A = \sin C \cos a\,\delta a,$$

and therefore $$\delta A \cot A = \delta a \cot a.$$

(4) Required the connexion between the small variations of a side and the adjacent angle (a, B).

We have $\qquad \cot C \sin B = \cot c \sin a$

proceeding as before we obtain

$$\cot C \cos B\, \delta B = \cot c \cos a\, \delta a + \cos B$$

therefore

$$(\cot C \cos B - \cos a \sin B)\, \delta B = (\cot c$$

therefore $$-\frac{\cos A}{\sin C}\, \delta B = \frac{\cos l}{\sin \iota}$$

therefore $$\delta B \cos A = -\delta a \cot b$$

130. Some more examples will be posed for solution at the end of this ch difficulty they are left for the exercise ol

131. *From any formula in Spheric the elements of a triangle, one of them l to deduce the corresponding formula in I*

Let α, β, γ be the lengths of the s radius of the sphere, so that $\frac{\alpha}{r}, \frac{\beta}{r}, \frac{\gamma}{r}$

of the sides of the triangle; expand

which occur in any proposed formul

respectively; then if we suppose r to the limiting form of the proposed forr Plane Trigonometry.

For example, in Art. 106, from the

$$\cos A = \frac{\cos a - \cos}{\sin b \text{ si}}$$

we deduce

$$\cos A = \frac{\beta^2 + \gamma^2 - \alpha^2}{2\beta\gamma} + \frac{\alpha^4 + \beta^4 + \gamma^4 - 2\alpha^2}{24}$$

now suppose r to become infinite; then ultimately

$$\cos A = \frac{\beta^2 + \gamma^2 - \alpha^2}{2\beta\gamma};$$

and this is the expression for the cosine of the angle of a plane triangle in terms of the sides.

Again, in Art. 110, from the formula

$$\frac{\sin A}{\sin B} = \frac{\sin a}{\sin b}$$

we deduce

$$\frac{\sin A}{\sin B} = \frac{\alpha}{\beta} + \frac{\alpha(\beta^2 - \alpha^2)}{6\beta r^2} + \ldots\ldots;$$

now suppose r to become infinite; then ultimately

$$\frac{\sin A}{\sin B} = \frac{\alpha}{\beta},$$

that is, in a plane triangle the sides are as the sines of the opposite angles.

EXAMPLES.

1. In a spherical triangle, if C and c remain constant while a and b receive the small increments δa and δb respectively, shew that

$$\frac{\delta a}{\sqrt{(1 - n^2 \sin^2 a)}} + \frac{\delta b}{\sqrt{(1 - n^2 \sin^2 b)}} = 0 \text{ where } n = \frac{\sin C}{\sin c}.$$

2. If C and c remain constant, and a small change be made in a, find the consequent changes in the other parts of the triangle. Find also the change in the area.

3. Supposing A and c to remain constant, prove the following equations, connecting the small variations of pairs of the other elements,

$\sin C\, \delta b = \sin a\, \delta B$, $\quad \delta b \sin C = -\delta C \tan a$, $\quad \delta a \tan C = \delta B \sin a$,

$\delta a \tan C = -\delta C \tan a$, $\quad \delta b \cos C = \delta a$, $\quad \delta B \cos a = -\delta C$.

4. Supposing b and c to rem
equations connecting the small
elements,

$$\delta B \tan C = \delta C \tan B,$$

$$\delta a = \sin c \sin B \delta A,$$

5. Supposing B and C to re
ing equations connecting the s
other elements,

$$\delta b \tan c = \delta c \tan b,$$

$$\delta A = \delta a \sin b \sin C,$$

6. From the formula sin $\frac{c}{2}$
duce the formula for the area
$\frac{a^2 \sin B \sin C}{2 \sin A}$, when the radius
creased.

7. If A and C are constant
quantity, shew that a will be in
as c is less or greater than a q

8. Two spherical triangles
differ slightly in position; prove

$$\cos ABb \cos BCc \cos CAa + \text{cc}$$

9. What formulæ in Plane T
Napier's Analogies; and what fro

10. From the formula

$$\cos \frac{c}{2} \cos \frac{A+B}{2} =$$

deduce the area of a plane triangl
of the angles.

11. What result is obtained
supposing the radius of the sphere

XII. POLYHEDRONS.

132. A polyhedron is a solid bounded by any number of plane rectilineal figures which are called its faces. A polyhedron is said to be *regular* when its faces are similar and equal regular polygons, and its solid angles equal to one another.

133. *If* S *be the number of solid angles in any polyhedron,* F *the number of its faces,* E *the number of its edges, then* S + F = E + 2.

Take any point within the polyhedron as centre, and describe a sphere of radius r, and draw lines from the centre to each of the angular points of the polyhedron; let the points in which these lines meet the surface of the sphere be joined by arcs of great circles, so that the surface of the sphere is divided into as many polygons as the polyhedron has faces.

Let s denote the sum of the angles of any one of these polygons, m the number of its sides; then the area of the polygon is $r^2\{s-(m-2)\pi\}$ by Art. 99. The sum of the areas of all the polygons is the surface of the sphere, that is, $4\pi r^2$. Hence since the number of the polygons is F, we obtain

$$4\pi = \Sigma s - \pi\Sigma m + 2F\pi.$$

Now Σs denotes the sum of all the angles of the polygons, and is therefore equal to $2\pi\times$ the number of solid angles, that is, to $2\pi S$; and Σm is equal to the number of all the sides of all the polygons, that is, to $2E$, since every edge gives rise to an arc which is common to two polygons. Therefore

$$4\pi = 2\pi S - 2\pi E + 2F\pi;$$

therefore $$S + F = E + 2.$$

134. *There can be only five regular polyhedrons.*

Let m be the number of sides in each face of a regular polyhedron, n the number of plane angles in each solid angle; then the entire number of plane angles is expressed by mF, or nS, or $2E$; thus

$$mF = nS = 2E, \text{ and } S + F = E + 2;$$

from these equations we obtain

$$S = \frac{4m}{2(m+n) - mn}, \quad E = \frac{2mn}{2(m+n) - mn}, \quad F = \frac{4n}{2(m+n) - mn}.$$

These expressions must be positive integers, we must therefore have $2(m+n)$ greater than mn; therefore

$$\frac{1}{m} + \frac{1}{n} \text{ must be greater than } \tfrac{1}{2};$$

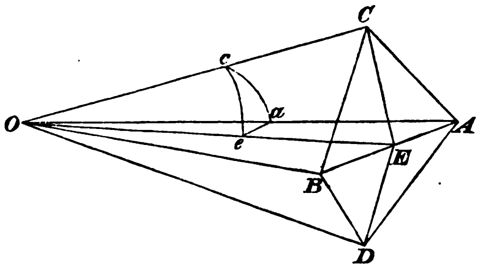
but n cannot be less than 3, so that $\frac{1}{n}$ cannot be greater than $\frac{1}{3}$, and therefore $\frac{1}{m}$ must be greater than $\frac{1}{6}$; and as m must be an integer and cannot be less than 3, the only admissible values of m are 3, 4, 5. It will be found on trial that the only values of m and n which satisfy all the necessary conditions are the following; each regular polyhedron derives its name from the number of its plane faces.

m	n	S	E	F	Name of regular Polyhedron.
3	3	4	6	4	Tetrahedron or regular Pyramid.
4	3	8	12	6	Hexahedron or Cube.
3	4	6	12	8	Octahedron.
5	3	20	30	12	Dodecahedron.
3	5	12	30	20	Icosahedron.

135. *The sum of all the plane angles which form the solid angles of any polyhedron is* $2(S-2)\pi$.

For if m denote the number of sides in any face of the polyhedron, the sum of the interior angles of that face is $(m-2)\pi$ by Euclid I. 32, Cor. 1. Hence the sum of all the interior angles of all the faces is $\Sigma(m-2)\pi$, that is $\Sigma m\pi - 2F\pi$, that is $2(E-F)\pi$, that is $2(S-2)\pi$.

136. *To find the inclination of two adjacent faces of a regular polyhedron.*



Let AB be the edge common to the two adjacent faces, C and D the centres of the faces; bisect AB in E and join CE and DE; CE and DE will be perpendicular to AB, and the angle CED is the angle of inclination of the two adjacent faces; we shall denote it by I. In the plane containing CE and DE draw CO and DO perpendicular to CE and DE respectively, and meeting in O; about O as centre describe a sphere meeting OA, OC, OE in a, c, e respectively, so that cae forms a spherical triangle. Since AB is perpendicular to CE and DE, it is perpendicular to the plane CED, therefore the plane AOB which contains AB is perpendicular to the plane CED; hence the angle cea of the spherical triangle is a right angle. Let m be the number of sides in each face of the polyhedron, n the number of the plane angles which form each solid angle. Then the angle $ace = ACE = \frac{2\pi}{2m} = \frac{\pi}{m}$; and the angle *cae* is half one of the *n* equal angles fo

that is, $cae = \frac{2\pi}{2n} = \frac{\pi}{n}$. From the right-

$$\cos cae = \cos cO$$

that is $$\cos\frac{\pi}{n} = \cos\left(\frac{\pi}{2}\right.$$

therefore $$\sin\frac{I}{2} = \frac{\cos\frac{\pi}{n}}{\sin\frac{\pi}{m}}$$

137. *To find the radii of the spheres of a regular polyhedron.*

Let the edge $AB = a$, let $OC = r$ the radius of the inscribed sphere, circumscribed sphere. Then

$$CE = AE \cot ACE = \frac{a}{2}\cot\frac{\pi}{m},$$

$$r = CE \tan CEO = CE \tan\frac{I}{2}$$

also $$r = R\cos aOc = R\cot eca \cot$$

therefore $R = r\tan\frac{\pi}{m}\tan\frac{\pi}{n} = \frac{a}{2}\tan\frac{I}{2}$ t

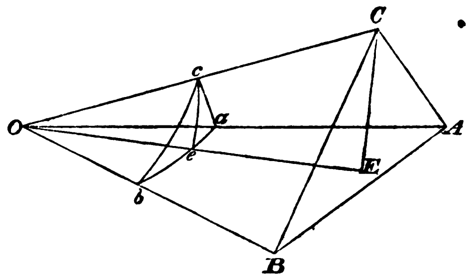
138. *To find the surface and volu*

The area of one face of the pol

therefore the surface of the polyhedron

Also the volume of the pyramid which has one face of the polyhedron for base and O for vertex is $\frac{r}{3} \cdot \frac{ma^2}{4} \cot \frac{\pi}{m}$, and therefore the volume of the polyhedron is $\frac{mFra^2}{12} \cot \frac{\pi}{m}$.

139. *To find the volume of a parallelopiped in terms of its edges and their inclinations to one another.*



Let the edges be $OA = a$, $OB = b$, $OC = c$; let the inclinations be $BOC = \alpha$, $COA = \beta$, $AOB = \gamma$. Draw CE perpendicular to the plane AOB meeting it in E. Describe a sphere with O as a centre, meeting OA, OB, OC, OE in a, b, c, e respectively.

The volume of the parallelopiped is equal to the product of its base and altitude $= ab \sin\gamma \,.\, CE = abc \sin\gamma \sin cOe$. The spherical triangle cae is right-angled at e; thus

$$\sin cOe = \sin cOa \sin cae = \sin\beta \sin cab,$$

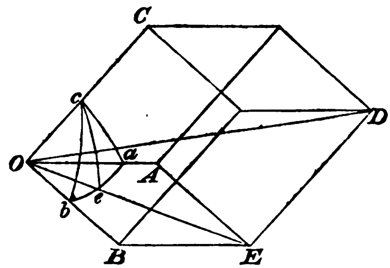
and from the spherical triangle cab

$$\sin cab = \frac{\sqrt{(1 - \cos^2\alpha - \cos^2\beta - \cos^2\gamma + 2\cos\alpha\cos\beta\cos\gamma)}}{\sin\beta\sin\gamma};$$

therefore the volume of the parallelopiped

$$= abc\sqrt{(1 - \cos^2\alpha - \cos^2\beta - \cos^2\gamma + 2\cos\alpha\cos\beta\cos\gamma)}.$$

140. *To find the diagonal of a parallelopiped in terms of the three edges which it meets and their inclinations to one another.*



Let the edges be $OA = a$, $OB = b$, $OC = c$; let the inclinations be $BOC = \alpha$, $COA = \beta$, $AOB = \gamma$. Let OD be the diagonal required, and let OE be the diagonal of the face OAB. Then

$$OD^2 = OE^2 + ED^2 + 2OE \,.\, ED \cos COE$$

$$= a^2 + b^2 + 2ab \cos\gamma + c^2 + 2cOE \cos COE.$$

Describe a sphere with O as centre meeting OA, OB, OC, OE in a, b, c, e respectively; then (see example 14, Chap. IV.)

$$\cos cOe = \frac{\cos cOb \sin aOe + \cos cOa \sin bOe}{\sin aOb}$$

$$= \frac{\cos\alpha \sin aOe + \cos\beta \sin bOe}{\sin\gamma};$$

therefore

$$OD^2 = a^2 + b^2 + c^2 + 2ab \cos\gamma + \frac{2cOE}{\sin\gamma}(\cos\alpha \sin aOe + \cos\beta \sin bOe),$$

and $\quad OE \sin aOe = b \sin\gamma, \qquad OE \sin bOe = a \sin\gamma;$

therefore $\quad OD^2 = a^2 + b^2 + c^2 + 2ab \cos\gamma + 2bc \cos\alpha + 2ca \cos\beta.$

141. *To find the volume of a tetrahedron.*

A tetrahedron is one-sixth of a parallelopiped which has the same altitude and its base double that of the tetrahedron; thus if the edges and their inclinations are given we can take one-sixth of the expression for the volume in Art. 139. The volume of a tetrahedron may also be expressed in terms of its six edges; for in the figure of Art. 139 let $BC = a'$, $CA = b'$, $AB = c'$; then .

$$\cos\alpha = \frac{b^2 + c^2 - a'^2}{2bc}, \quad \cos\beta = \frac{c^2 + a^2 - b'^2}{2ca}, \quad \cos\gamma = \frac{a^2 + b^2 - c'^2}{2ab},$$

and if these values are substituted for $\cos\alpha$, $\cos\beta$, and $\cos\gamma$ in the expression obtained in Art. 139, the volume of the tetrahedron will be expressed in terms of its six edges.

EXAMPLES.

1. If I denote the inclination of two adjacent faces of a regular polyhedron, shew that $\cos I = \frac{1}{3}$ in the tetrahedron, $= 0$ in the cube, $= -\frac{1}{3}$ in the octahedron, $= -\frac{1}{5}\sqrt{5}$ in the dodecahedron, and $= -\frac{1}{3}\sqrt{5}$ in the icosahedron.

2. With the notation of Art. 136, shew that the radius of the sphere which touches one face of a regular polyhedron and all the adjacent faces produced is $\frac{1}{2} a \cot\frac{\pi}{m} \cot\frac{1}{2}I$.

3. A sphere touches one face of a regular tetrahedron and the other three faces produced; find its radius.

4. If a and b are the radii of the spheres inscribed in and described about a regular tetrahedron, shew that $b = 3a$.

5. If a is the radius of a sphere inscribed in a regular tetrahedron, and R the radius of the sphere which touches the edges, shew that $R^2 = 3a^2$.

6. If a is the radius of a sphere inscribed in a regular tetrahedron, and R' the radius of the sphere which touches one face and the others produced, shew that $R' = 2a$.

7. If a cube and an octahedron be described about a given sphere, the sphere described about these polyhedrons will be the *same*; and conversely.

8. If a dodecahedron and an icosahedron be described about a given sphere, the sphere described about these polyhedrons will be the same; and conversely.

9. A regular tetrahedron and a regular octahedron are inscribed in the same sphere; compare the radii of the spheres which can be inscribed in the two solids.

10. The sum of the squares of the four diagonals of a parallelopiped is equal to four times the sum of the squares of the edges.

11. If with each angular point of any parallelopiped as centres equal spheres be described, the sum of the intercepted portions of the parallelopiped will be equal in volume to one of the spheres.

12. A regular octahedron is inscribed in a cube so that the corners of the octahedron are in the centres of the faces of the cube; prove that the volume of the cube is six times that of the octahedron.

13. It is not possible to fill any given space with a number of regular polyhedrons of the same kind, except cubes; but this may be done by means of tetrahedrons and octahedrons which have equal faces, by using twice as many of the former as of the latter.

14. A spherical triangle is formed on the surface of a sphere of radius ρ; its angular points are joined, forming thus a pyramid with the lines joining them with the centres; shew that the volume of the pyramid is

$$\tfrac{1}{3}\rho^3 \sqrt{(\tan r \tan r_1 \tan r_2 \tan r_3)}$$

where r, r_1, r_2, r_3 are the radii of the inscribed and escribed circles of the triangle.

15. The angular points of a regular tetrahedron inscribed in a sphere of radius r being taken as poles, four equal small circles of the sphere are described, so that each circle touches the other

three. Shew that the area of the surface bounded by each circle is $2\pi r^2\left(1-\frac{1}{\sqrt{3}}\right)$.

16. If O be any point within a spherical triangle ABC, the product of the sines of any two sides and the sine of the included angle

$$= \sin AO \sin BO \sin CO \left\{\cot AO \sin BOC + \cot BO \sin COA + \cot CO \sin AOB\right\}.$$

XIII. MISCELLANEOUS PROPOSITIONS.

142. *To find the equation to a small circle of the sphere.*

Let O be the pole of a small circle, S a fixed point on the sphere, SX a fixed great circle of the sphere. Let $OS = a$, $OSX = \beta$; then the position of O is determined by means of these angular co-ordinates a and β. Let P be any point on the circumference of the small circle, $PS = \theta$, $PSX = \phi$, so that θ and ϕ are the angular co-ordinates of P. Let $OP = r$. Then from the triangle OSP

$$\cos r = \cos a \cos \theta + \sin a \sin \theta \cos(\phi - \beta) \quad \ldots\ldots\ldots\ldots(1);$$

this gives a relation between the angular co-ordinates of any point on the circumference of the circle.

If the circle be a great circle then $r = \frac{\pi}{2}$; thus the equation becomes

$$0 = \cos a \cos \theta + \sin a \sin \theta \cos(\phi - \beta) \quad \ldots\ldots\ldots(2).$$

It will be observed that the angular co-ordinates here used are analogous to the *latitude* and *longitude* which serve to determine the positions of places on the Earth's surface; θ is the *complement of the latitude* and ϕ is the *longitude*.

143. *To find the locus of the vertex of a spherical triangle of given base and area.*

Let AB be the given base, $=c$ su[redacted]
Since the area is given the spherical [redacted]
it by E; then by Art. 103,

$$\cot \tfrac{1}{2} E = \cot \tfrac{1}{2} \theta \cot \tfrac{1}{2} c \cos$$ [redacted]

therefore $\quad \sin (\phi - \tfrac{1}{2} E) = \cot \tfrac{1}{2} \theta \cot$ [redacted]

therefore $\quad 2 \cot \tfrac{1}{2} c \sin \tfrac{1}{2} E \cos^2 \dfrac{\theta}{2} = \sin$ [redacted]

therefore

$$\cos \theta \cot \tfrac{1}{2} c \sin \tfrac{1}{2} E + \sin \theta \cos \left(\phi - \tfrac{1}{2} E\right.$$ [redacted]

Comparing this with equation (1) of [redacted]
see that the required locus is a circle. I[redacted]
co-ordinates of its pole, we have

$$\tan \alpha = \frac{1}{\cot \frac{1}{2} c \sin \frac{1}{2} E} =$$ [redacted]

$$\beta = \tfrac{1}{2} E - \frac{\pi}{2}.$$

It may be presumed from symmet[redacted]
circle is in the great circle which bis[redacted]
and this presumption is easily verifie[redacted]
that great circle is

$$0 = \cos \theta \cos \left(\frac{\pi}{2} - \frac{c}{2}\right) + \sin \theta \sin \left(\frac{\pi}{2}\right.$$ [redacted]

and the values $\theta = \alpha$, $\phi = \beta$ satisfy thi[redacted]

144. *To find the angular distance* [redacted]
inscribed and circumscribed circles of a t[redacted]

Let P denote the pole of the inscrib[redacted]
of the circumscribed circle of a triangl[redacted]
$PAB = \tfrac{1}{2} A$, and by Art. 92 $QAB =$ [redacted]
$= \cos \tfrac{1}{2}(B - C)$; and

$$\cos PQ = \cos PA \cos QA + \sin PA$$ [redacted]

Now, by Art. 62 (see figure of Art. 89),

$$\cos PA = \cos PE \cos AE = \cos r \cos (s-a),$$

$$\sin PA = \frac{\sin PE}{\sin PAE} = \frac{\sin r}{\sin \frac{1}{2}A};$$

thus

$$\cos PQ = \cos R \cos r \cos (s-a) + \sin R \sin r \cos \tfrac{1}{2}(B-C) \operatorname{cosec} \tfrac{1}{2}A$$

$$= \cos R \cos r \cos (s-a) + \sin R \sin r \sin \tfrac{1}{2}(b+c) \operatorname{cosec} \tfrac{1}{2}a,$$

(Art. 54);

therefore $$\frac{\cos PQ}{\cos R \sin r} = \cot r \cos (s-a) + \tan R \sin \tfrac{1}{2}(b+c) \operatorname{cosec} \tfrac{1}{2}a.$$

Now $$\cot r = \frac{\sin s}{n}, \quad \tan R = \frac{2 \sin \frac{1}{2}a \sin \frac{1}{2}b \sin \frac{1}{2}c}{n},$$

therefore $$\frac{\cos PQ}{\cos R \sin r} = \frac{1}{n}\left\{\sin s \cos (s-a) + 2 \sin \tfrac{1}{2}(b+c) \sin \tfrac{1}{2}b \sin \tfrac{1}{2}c\right\}$$

$$= \frac{1}{2n}(\sin a + \sin b + \sin c).$$

Hence $$\left(\frac{\cos PQ}{\cos R \sin r}\right)^2 - 1 = \frac{1}{4n^2}(\sin a + \sin b + \sin c)^2 - 1$$

$$= (\cot r + \tan R)^2 \text{ by (Art. 94)};$$

therefore $$\cos^2 PQ = \cos^2 R \sin^2 r + \cos^2 (R-r),$$

and $$\sin^2 PQ = \sin^2 (R-r) - \cos^2 R \sin^2 r.$$

145. *To find the angular distance between the pole of the circumscribed circle and the pole of one of the escribed circles of a triangle.*

Let Q denote the pole of the circumscribed circle, and Q_1 the pole of the escribed circle opposite to the angle A. Then it may be shewn that $QBQ_1 = \frac{1}{2}\pi + \frac{1}{2}(C-A)$, and

$$\cos QQ_1 = \cos R \cos r_1 \cos (s-c) - \sin R \sin r_1 \sin \tfrac{1}{2}(C-A) \sec \tfrac{1}{2}B$$

$$= \cos R \cos r_1 \cos (s-c) - \sin R \sin r_1 \sin \tfrac{1}{2}(c-a) \operatorname{cosec} \tfrac{1}{2}b.$$

Therefore

$$\frac{\cos QQ_1}{\sin r_1 \cos R} = \cot r_1 \cos (s-c) - \tan R \sin \tfrac{1}{2}(c-a) \operatorname{cosec} \tfrac{1}{2}b;$$

by reducing as in the preceding articl
the last equation becomes

$$\frac{1}{2n}(\sin b + \sin c -$$

hence $\left(\frac{\cos QQ_1}{\cos R \sin r_1}\right)^2 - 1 = (\tan$

therefore $\cos^2 QQ_1 = \cos^2 .$

and $\sin^2 QQ_1 = \sin^2 ($

146. *The arc which passes thro*
sides of any triangle upon a given ba
in a fixed point, the distance of which
base is a quadrant.

Let ABC be any triangle, E the
the middle point of AB; let the arc
produced meet BC produced in Q. T

$$\frac{\sin BQ}{\sin BF} = \frac{\sin BFQ}{\sin BQF}, \qquad \frac{\text{si}}{\text{si}}$$

therefore $\frac{\sin BQ}{\sin AQ} = \frac{\sin A(}{\sin B(}$

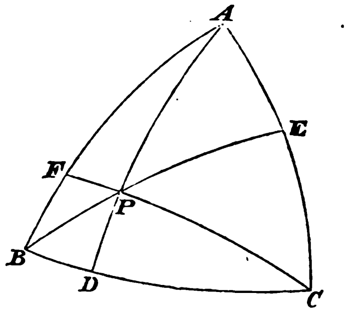
similarly $\frac{\sin CQ}{\sin AQ} = \frac{\sin A(}{\sin C(}$

therefore $\sin BQ = \sin CQ$; theref

Hence if D be the middle point

$$DQ = \tfrac{1}{2}(BQ + C($$

147. *If three arcs be drawn fr*
triangle through any point to meet th
of the sines of the alternate segments o



Let P be any point, and let arcs be drawn from the angles A, B, C passing through P and meeting the opposite sides in D, E, F. Then

$$\frac{\sin BD}{\sin BP} = \frac{\sin BPD}{\sin BDP}, \qquad \frac{\sin CD}{\sin CP} = \frac{\sin CPD}{\sin CDP},$$

therefore
$$\frac{\sin BD}{\sin CD} = \frac{\sin BPD}{\sin CPD} \frac{\sin BP}{\sin CP}.$$

Similar expressions may be found for $\dfrac{\sin CE}{\sin AE}$ and $\dfrac{\sin AF}{\sin BF}$; and hence it follows obviously that

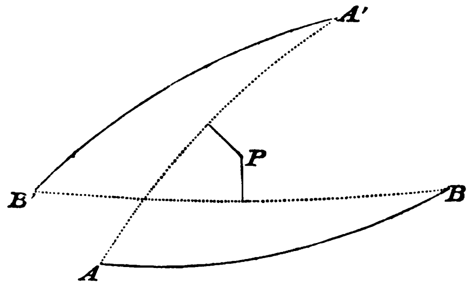
$$\frac{\sin BD}{\sin CD} \frac{\sin CE}{\sin AE} \frac{\sin AF}{\sin BF} = 1;$$

therefore $\quad \sin BD \sin CE \sin AF = \sin CD \sin AE \sin BF.$

148. Conversely, when the points D, E, F in the sides of a spherical triangle are such that the relation given in the preceding article holds, the arcs which join these points with the opposite angles respectively *pass through a common point.* Hence the following propositions may be established:—the perpendiculars from the angles of a spherical triangle upon the opposite sides meet in a point; the lines which bisect the angles of a spherical triangle meet in a point; the lines which join the angles of a spherical triangle with the middle points of the opposite sides meet in a point; the lines which join the angles of a spherical

triangle with the points where the inscribed circle touches the opposite sides respectively meet in a point.

149. *If* AB *and* A′B′ *be any two equal arcs, and the arcs* AA′ *and* BB′ *be bisected at right angles by arcs meeting in* P, *then* AB *and* A′B′ *subtend equal angles at P.*



For $PA = PA'$ and $PB = PB'$; hence the sides of the triangle PAB are respectively equal to those of $PA'B'$; therefore the angle $APB =$ the angle $A'PB'$.

This simple proposition has an important application to the motion of a rigid body of which one point is fixed. For conceive a sphere capable of motion round its centre which is fixed; then it appears from this proposition that any two fixed points on the sphere, as A and B, can be brought into any other positions, as A' and B', by rotation round an axis passing through the centre of the sphere and a certain point P. Hence it may be inferred that any change of position in a rigid body, of which one point is fixed, may be effected by rotation round some axis through the fixed point.

(De Morgan's *Differential and Integral Calculus*, page 489).

MISCELLANEOUS EXAMPLES.

1. Find the locus of the vertices of all right-angled spherical triangles having the same hypothenuse; and from the equation obtained, prove that the locus is a circle when the radius of the sphere is infinite.

2. AB is an arc of a great circle on the surface of a sphere, C its middle point; shew that the locus of the point P such that the angle APC = the angle BPC consists of two great circles at right angles to one another. Explain this when the triangle becomes plane.

3. A point P being taken on the surface of a sphere, let ψ and ψ' denote its spherical distances from two given points; if $m \cos \psi + m' \cos \psi'$ is constant where m and m' are constant, shew that the locus of P is a circle.

4. On a given arc of a sphere, spherical triangles of equal area are described; shew that the locus of the angular point opposite to the given arc is defined by the equation

$$\tan^{-1}\left\{\frac{\tan(a+\phi)}{\sin\theta}\right\} + \tan^{-1}\left\{\frac{\tan(a-\phi)}{\sin\theta}\right\} + \tan^{-1}\left\{\frac{\tan\theta}{\sin(a+\phi)}\right\} + \tan^{-1}\left\{\frac{\tan\theta}{\sin(a-\phi)}\right\} = \beta,$$

where $2a$ is the length of the given arc, θ the arc of the great circle drawn from any point P in the locus perpendicular to the given arc, ϕ the inclination of the great circle on which θ is measured to the great circle bisecting the given arc at right angles, and β a constant.

5. If θ, ϕ, ψ denote the distances from the angles A, B, C respectively of the point of intersection of arcs bisecting the angles of the spherical triangle ABC, shew that

$$\cos\theta \sin(b-c) + \cos\phi \sin(c-a) + \cos\psi \sin(a-b) = 0.$$

6. If A', B', C' be the poles of the sides BC, CA, AB of a spherical triangle ABC, shew that the great circles AA', BB', CC' meet in a point P, such that

$$\cos PA \cos BC = \cos PB \cos CA = \cos PC \cos AB.$$

7. If O be the point of intersection of arcs AD, BE, CF drawn from the angles of a triangle perpendicular to the opposite

sides and meeting them in D, E, F res

$$\frac{\tan AD}{\tan OD}, \quad \frac{\tan BE}{\tan OE},$$

are respectively equal to

$$1+\frac{\cos A}{\cos B \cos C}, \quad 1+\frac{\cos B}{\cos A \cos}$$

8. If p, q, r be the arcs of gre angles of a triangle perpendicular to (β, β'), (γ, γ') the segments into whi shew that

$$\tan \alpha \tan \alpha' = \tan \beta \tan \beta'$$

and
$$\frac{\cos p}{\cos \alpha \cos \alpha'} = \frac{\cos q}{\cos \beta \cos \beta'}$$

9. In a spherical triangle if arcs the middle points of the opposite sic parts of the one which bisects the side

$$\frac{\sin \alpha}{\sin \alpha'} = 2 \cos \frac{a}{2}$$

10. The arc of a great circle bise spherical triangle cuts BC produced in

$$\cos AQ \sin \frac{a}{2} = \sin \frac{c-l}{2}$$

11. If $ABCD$ be a spherical qua sides AB, CD when produced meet in the ratio of the sines of the arcs drawn the diagonals of the quadrilateral is th from F.

12. If $ABCD$ be a spherical qua DC are produced to meet in P, and diagonals AC, BD intersect in R, th $\sin AB \sin CD \cos P \sim \sin AD \sin BC$ c

13. If the arcs joining the extren rical triangle with the middle points equal, the triangle is isosceles.

14. The arc AP of a circle of the same radius as the sphere is equal to the greater of two sides of a spherical triangle, and the arc AQ taken in the same direction is equal to the less; the sine PM of AP is divided in E, so that $\frac{EM}{PM}$ = the natural cosine of the angle included by the two sides, and EZ is drawn parallel to the tangent to the circle at Q. Shew that the remaining side of the spherical triangle is equal to the arc QPZ.

15. If through any point P within a spherical triangle ABC great circles be drawn from the angular points A, B, C to meet the opposite sides in a, b, c respectively, prove that

$$\frac{\sin Pa \cos PA}{\sin Aa} + \frac{\sin Pb \cos PB}{\sin Bb} + \frac{\sin Pc \cos PC}{\sin Cc} = 1.$$

16. A and B are two places on the Earth's surface on the same side of the equator, A being further from the equator than B. If the bearing of A from B be more nearly due East than it is from any other place in the same latitude as B, what is the bearing of B from A?

17. If the tangent of the radius of the circle described about a spherical triangle is equal to twice the tangent of the radius of the circle inscribed in the triangle, the triangle is equilateral.

18. From the result given in example 18 of Chapter v. infer the possibility of a regular dodecahedron.

THE END.

CAMBRIDGE: PRINTED BY C. J. CLAY, M.A. AT THE UNIVERSITY PRESS.

SELECT LIST OF

New Works and New Editions.

PUBLISHED BY

MACMILLAN AND CO.

CAMBRIDGE,

AND 23, HENRIETTA STREET, COVENT GARDEN, LONDON.

LIFE OF JOHN MILTON,

Narrated in connexion with the Political, Ecclesiastical, and Literary History of his Time. By DAVID MASSON, M.A. Professor of English Literature in University College, London. VOL. I. 8vo. With Two Portraits. 18*s.*

"... *The author has endeavoured not only to sketch Milton such as he was, the representative poet and literary man of his day, but to exhibit him 'in connexion with the more notable phenomena in which his life was cast, its state politics, its ecclesiastical variations, its literature and speculative thought.' ... Such a historical area does Mr. Masson's Life of Milton occupy. There is scarce a page which does not bear witness of careful antiquarian research and minute as well as extensive reading. The biographer has large acquaintance with his period, deep love for his author.*"—ATHENÆUM.

Eighth Thousand.

SCOURING OF THE WHITE HORSE.

By the Author of "TOM BROWN'S SCHOOL DAYS." With numerous Illustrations by RICHARD DOYLE. Imp. 16mo. printed on toned paper, gilt leaves. 8*s.* 6*d.*

"*The execution is excellent.... Like Tom Brown's School Days, the White Horse gives the reader a feeling of gratitude and personal esteem towards the author, The author could not have a better style, nor a better temper, nor a more excellent artist than Mr. Doyle to adorn his book.*"—SATURDAY REVIEW.

"*There is in it a high tone of earnest purpose, which engages our strongest interest and to which we most heartily say God speed it! ... With its kindly feeling, its neighbourly and hearty spirit, its sympathy with the homes of England and those ties which used to bind Englishmen to them and its cheerful Christianity.*" —PRESS.

26. 7. 59.
1,000

FOUR MONTHS IN ALGERIA:

WITH A VISIT TO CARTHAGE.

By J. W. Blakesley, B.D., Vicar of Ware, Herts: sometime Fellow and Tutor of Trinity College, Cambridge. With Maps and Illustrations after Photographs. 8vo. cloth. 14s.

"*Rich in delightful descriptions of scenery, and in narrative of personal adventure and observation. . . . When the circumstances of its temporary claims on attention have passed away, it will continue to be prized by the antiquarian and scholar for its solid and valuable information, and by the lovers of books on foreign lands and peoples for the brightness of its pictures and its amusing notes of travel.*"—Nonconformist.

AGNES HOPETOUN'S SCHOOLS AND HOLIDAYS:

THE EXPERIENCES OF A LITTLE GIRL.

By Mrs. Oliphant, Author of "Mrs. Margaret Maitland," &c. Royal 16mo. bound in extra cloth. 6s.

"*One of Mrs. Oliphant's gentle, thoughtful stories. . . . Described with exquisite reality and thorough appreciation of childish ethics, proving that the gifted writer has the same power over the hearts of the young as she exercised over those of their elders . . . teaching them pure and good lessons.*"—John Bull.

DAYS OF OLD:

STORIES FROM OLD ENGLISH HISTORY FOR THE YOUNG, ON THE DRUIDS, THE ANGLO-SAXONS, AND THE CRUSADES.

By the Author of "Ruth and her Friends." Royal 16mo. printed on toned paper. 5s.

"*A delightful little book, full of interest and instruction . . . fine feeling, dramatic weight, and descriptive power in the stories. . . . They are valuable as throwing a good deal of light upon English history, bringing rapidly out the manners and customs, the social and political conditions of our British and Anglo-Saxon ancestors, and the moral always of a pure and noble kind.*"—Literary Gazette.

THE TWELVE FOUNDATIONS,

AND OTHER POEMS.

By the Rev. H. C. ADAMS, M.A, late Fellow of Magdalen College, Oxford, Author of "Sivan the Sleeper," &c. Royal 16mo. 5*s.*

"*The production of a truly poetic mind and heart. . . . Some of the stanzas are exquisitely beautiful, and all are touchingly spiritual and devout.*"—FREEMAN.

THE ILIAD OF HOMER.

TRANSLATED INTO ENGLISH VERSE.

By I. C. WRIGHT, M.A., Translator of "Dante," late Fellow of Magdalen College, Oxford. Books I.—VI. Crown 8vo. 5*s.*

"*We know of no edition of the 'sovran poet' from which an English reader can derive on the whole so complete an impression of the immortal Epos.*"—DAILY NEWS.

LAYS OF MIDDLE AGE,

AND OTHER POEMS.

By JAMES HEDDERWICK. Fcap. 8vo. cloth. 5*s.*

"*We cannot understand how it is that Mr. Hedderwick's poetry is not a thousand times better known. It needs only to be known to be appreciated. There is something delightful in its calm, healthful pathos and beauty.*"—FRASER'S MAGAZINE, Feb. 1857.

A STORY FOR GIRLS.

Ruth and Her Friends. With a Frontispiece. Fcap. 8vo. cloth. Second Edition. 5*s.*

"*It is a book which girls will read with avidity, and can hardly fail to profit by.*"—LITERARY CHURCHMAN.

"*Seldom, if ever, have more intellectual power and healthful sentiment gone to the production of a story for girls; and we wish all the girls in the land had the opportunity of reading it.*"—NONCONFORMIST.

BY THE LATE GEORGE BRIMLEY, M.A.

Librarian of Trinity College, Cambridge.

Essays. Edited by WILLIAM GEORGE CLARK, M.A., Fellow and Tutor of Trinity College, and Public Orator in the University of Cambridge. With Portrait. Crown 8vo. 7*s.* 6*d.*

CONTENTS.

I. TENNYSON'S POEMS.
II. WORDSWORTH'S POEMS.
III. POETRY AND CRITICISM.
IV. ANGEL IN THE HOUSE.
V. CARLYLE'S LIFE OF STERLING.
VI. ESMOND.
VII. MY NOVEL.
VIII. BLEAK HOUSE.
IX. WESTWARD HO!
X. WILSON'S NOCTES.
XI. COMTE'S POSITIVE PHILOSOPHY.

"*One of the most delightful and precious volumes of criticism that has appeared in these days. . . . To every cultivated reader they will disclose the wonderful clearness of perception, the delicacy of feeling, the pure taste, and the remarkably firm and decisive judgment which are the characteristics of all Mr. Brimley's writings on subjects that really penetrated and fully possessed his nature.*"—NONCONFORMIST.

BY G. M. HUMPHRY, M.B. Cantab. F.R.C.S.,

Surgeon to Addenbrooke's Hospital, Lecturer on Anatomy and Surgery in the Medical School, Cambridge.

A Treatise on the Human Skeleton, including the Joints. With Sixty Illustrations drawn from Nature.

Medium 8vo. cloth, 1*l.* 8*s.*

"*It does not consist of a mere detailed account of* dry *bones, but a beautifully written description of their development, growth, structure, union by joint, position in the human economy, varieties in the lower animals, and uses in standing, walking, and running, and leads gradually on to the last most interesting chapter on The Homology.*"—DUBLIN MEDICAL JOURNAL.

MEMOIR OF THE REV. GEORGE WAGNER,

Late of St. Stephen's, Brighton.

By J. N. SIMPKINSON, M.A., Rector of Brington, Northampton. Second Edition. Crown 8vo. cloth, 9*s.*

"*A deeply interesting picture of the life of one of a class of men who are indeed the salt of this land.*"—MORNING HERALD.

BY JOHN MALCOLM LUDLOW,

Barrister-at-Law.

British India, Its Races, and its History, down to the Mutinies of 1857. A Series of Lectures.

2 vols. fcap. 8vo. cloth, 9*s*.

" *The best historical Indian manual existing, one that ought to be in the hands of every man who writes, speaks, or votes on the Indian question.*"—EXAMINER.

" *The best elementary work on the History of India.*"—HOMEWARD MAIL.

THE REPUBLIC OF PLATO.

A New Translation into English. By J. LL. DAVIES, M.A., and D. J. VAUGHAN, M.A., Fellows of Trinity College, Cambridge. SECOND EDITION. 8vo. cloth, 10*s*. 6*d*.

" *So eloquent and correct a version will, we trust, induce many to become students of the Republic. . . The whole book is scholarlike and able.*"—GUARDIAN.

" *Free, nervous, idiomatic English, such as will fascinate the reader.*"—NONCONFORMIST.

BY GEORGE WILSON, M.D., F.R.S.E.,

Regius Professor of Technology in the University of Edinburgh; and Director of the Industrial Museum of Scotland.

Sixth Thousand.

1. The Five Gateways of Knowledge. A Popular Work on the Five Senses. In fcap. 8vo. cloth, with gilt leaves, 2*s*. 6*d*. PEOPLE'S EDITION, in ornamental stiff covers, 1*s*.

" *Dr. Wilson unites poetic with scientific faculty, and this union gives a charm to all he writes. In the little volume before us he has described the five senses in language so popular that a child may comprehend the meaning, so suggestive that philosophers will read it with pleasure.*"—LEADER.

2. The Progress of the Telegraph. Fcap. 8vo. 1*s*.

" *Most interesting and instructive . . . at once scientific and popular, religious and technical; a worthy companion to the 'Gateways of Knowledge.'*"—LITERARY CHURCHMAN.

PUBLISHED BY M

THE

WILLIAM ARC

Late Professor of Moral Phi

FIVE VOLUMES 8*vo*. *UNI*

"*A man of glowing genius and*
these five brilliant volumes."—E

SOLD SEPARA

1. Sermons, Doctrinal
Edited by the Very Rev. T
with a Memoir and Portr

"*Present a richer combination of*
any we have met with in any livi

2. Sermons, Doctrinal
Edited by J. A. Jeremie,
the University of Cambrid

"*They are marked by the same*
richness of imagery and illustrat
the same depth and fervour of
guished the preceding Series an
our theological literature."—Fro

3. Letters on Romanism
Development. Edited by th
Dean of Down. Second
deacon Hardwick.

"*Deserve to be considered the*
table energy and power of conce

4. Lectures on the His
Edited from the Author's
worth Thompson, M.A.
University of Cambridge.

"*Of the dialectic and physics of*
accurate, and popular, with whic
the French, and incomparably m
departments of the Platonic phil

THIRD EDITION.

Lectures to Ladies on Practical Subjects. Crown 8vo. 7*s*. 6*d*.

By F. D. Maurice, Charles Kingsley, J. Ll. Davies, Archdeacon Allen, Dean Trench, Professor Brewer, Dr. George Johnson, Dr. Sieveking, Dr. Chambers, F. J. Stephen, Esq. and Tom Taylor, Esq.

Contents:—Plan of Female Colleges—The College and the Hospital—The Country Parish—Overwork and Anxiety—Dispensaries—District Visiting—Influence of Occupation on Health—Law as it affects the Poor—Everyday Work of Ladies—Teaching by Words—Sanitary Law—Workhouse Visiting.

"*We scarcely know a volume containing more sterling good sense, or a finer expression of modern intelligence on social subjects.*"—Chambers' Journal.

PARAGUAY, BRAZIL, AND THE PLATE.

By Charles Mansfield, M.A., Clare College, Cambridge. With a Life by Charles Kingsley, Rector of Eversley. Post 8vo. With Map and numerous Illustrations. 12*s*. 6*d*.

"*An interesting and instructive volume.*"—Morning Post.

BY THE RIGHT REV. JOHN WILLIAM COLENSO, D.D.,

Lord Bishop of Natal, formerly Fellow of St. John's College, Cambridge.

1. **The Colony of Natal.** A Journal of Ten Weeks' Tour of Visitation among the Colonists and Zulu Kaffirs of Natal. With four Lithographs and a Map. Fcap. 8vo. cloth, 5*s*.
2. **Village Sermons.** Second Edition. Fcap. 8vo. cloth, 2*s*. 6*d*.
3. **Companion to the Holy Communion:** containing the Service, with Select Readings from the Writings of Mr. MAURICE. Fine Edition, rubricated and bound in morocco antique, gilt edges, 6*s*.; or in cloth, red edges, 2*s*. 6*d*.; common paper, limp cloth, 1*s*.

BY CHARLES ANTHONY SWAINSON, M.A.

Principal of the Theological College, and Prebendary of Chichester.

The Creeds of The Church. In their Relations to the Word of God and to the Conscience of the Christian. 8vo. cloth, 9*s*.

Contents:—I. Faith in God.—II. Exercise of our Reason.—III. Origin and Authority of Creeds.—IV. Inductive Proof of the Creeds.—V. Continual Guidance of the Spirit.—VI. Test and Application of Scripture.—VII. Private Judgment.—VIII. Strengthening of the *Judgment* and the Preparation for Controversy. With an Appendix.

PUBLISHED BY M

BY JULIUS CH.

Sometime Archdeacon of Lewes, Rector
Queen, and formerly Fellow and

NINE VOLS. 8vo. UNIFOR

1. Charges to the Cler
Lewes. During 1840 to
Events affecting the Churc
duction, explanatory of h
ference to the Parties whicl

2. Miscellaneous Pamphl
Questions agitated in the Cl

3. Vindication of Luther
Assailants. Second Edit

4. The Mission of the C
Edition.

5. The Victory of Faith.

6. Parish Sermons. Secon

7. Sermons preacht on Pa

The two following books are included a
separately for

Charges to the Clergy
Lewes. Delivered in th
before published. With a
position in the Church, with
it.

The Contest with Rome.
With Notes, especially in ans
of Catholics in England. S

BY JOHN McLEOD CAMPBELL,

Formerly Minister of Row.

The Nature of the Atonement, and its Relation to Remission of Sins and Eternal Life.

8vo. cloth, 10*s*. 6*d*.

"*This is a remarkable book, as indicating the mode in which a devout and intellectual mind has found its way, almost unassisted, out of the extreme Lutheran and Calvinistic views of the Atonement into a healthier atmosphere of doctrine. . . . We cannot assent to all the positions laid down by this writer, but he is entitled to be spoken respectfully of, both because of his evident earnestness and reality, and the tender mode in which he deals with the opinions of others from whom he feels compelled to differ.*"—LITERARY CHURCHMAN.

BY THE RIGHT REV. G. E. LYNCH COTTON, D.D.,

Lord Bishop of Calcutta and Metropolitan of India.

Sermons and Addresses delivered in Marlborough College, during Six Years.

Crown 8vo. cloth, price 10*s*. 6*d*.

"*We can heartily recommend this volume as a most suitable present for a youth, or for family reading, wherever there are young persons, the teaching of these discourses will be admirable.*"—LITERARY CHURCHMAN.

Sermons: Chiefly connected with Public Events in 1854.

Fcap. 8vo. cloth, 3*s*.

"*A volume of which we can speak with high admiration.*"
CHRISTIAN REMEMBRANCER.

BY JOHN HAMILTON, Esq. (of St. Ernan's,) M.A.,

St. John's College, Cambridge.

On Truth and Error: Thoughts, in Prose and Verse, on the Principles of Truth, and the Causes and Effects of Error.

Crown 8vo. bound in cloth, with red leaves, 10*s*. 6*d*.

"*A very genuine, thoughtful, and interesting book, the work of a man of honest mind and pure heart; one who has felt the pressure of religious difficulties, who has thought for himself on the matters of which he doubted, and who has patiently and piously worked his way to conclusions which he now reverently but fearlessly utters to the world.*"—NONCONFORMIST.

BY CHARLES KINGSLEY,

Rector of Eversley.

1. Two Years Ago. Second Edition.
3 vols. crown 8vo. cloth, £1 11*s.* 6*d.*

2. The Heroes: Greek Fairy Tales for my Children.
New and Cheaper Edition, with Eight Illustrations. Royal 16mo. beautifully printed on toned paper, gilt edges, 5*s.*

"*We doubt not they will be read by many a youth with an enchained interest almost as strong as the links which bound Andromeda to her rock.*"—BRITISH QUARTERLY.

3. "Westward Ho!" or the Voyages and Adventures of Sir Amyas Leigh, Knight, of Borrough, in the County of Devon, in the reign of Her most Glorious Majesty Queen Elizabeth. Third Edition. Crown 8vo. cloth, 7*s.* 6*d.*

4. Glaucus; or, the Wonders of the Shore. A Companion for the Sea-side. Containing Coloured Plates of the Objects mentioned in the Work. Third Edition. Fcap. 8vo. gilt leaves, 6*s.* 6*d.*

*** The Illustrated Companion may be had separately, price 3*s.* 6*d.*

"*Its pages sparkle with life, they open up a thousand sources of unanticipated pleasure, and combine amusement with instruction in a very happy and unwonted degree.*"—ECLECTIC REVIEW.

5. Phaethon; or, Loose Thoughts for Loose Thinkers.
Third Edition. Crown 8vo. boards, 2*s.*

"*Its suggestions may meet half way many a latent doubt, and, like a light breeze, lift from the soul clouds that are gathering heavily, and threatening to settle down in wintry gloom on the summer of many a fair and promising young life.*" —SPECTATOR.

6. Alexandria and Her Schools. Four Lectures delivered at the Philosophical Institution, Edinburgh. With a Preface.
Crown 8vo. cloth, 5*s.*

"*A series of brilliant biographical and literary sketches, interspersed with comments of the closest modern, or rather universal application.*"—SPECTATOR.

BY THE RIGHT REV. GEORGE AUGUSTUS SELWYN, D.D.,

Lord Bishop of New Zealand, formerly Fellow of St. John's College, Cambridge.

The Work of Christ in the World. Third Edition.
Published for the benefit of the New Zealand Church Fund.
Crown 8vo. 2*s.*

BY ARCHDEACON HARDWICK.

Christ and other Masters: A Historical Inquiry into some of the chief Parallelisms and Contrasts between Christianity and the Religious Systems of the Ancient World.

Part I. Introduction. Part II. Religions of India. Part III. Religions of China, America, and Oceanica. Part IV. Religions of Egypt and Medo-Persia. In 8vo. cloth, 7*s.* 6*d.* each.

"*Never was so difficult and complicated a subject as the history of Pagan religion handled so ably, and at the same time rendered so lucid and attractive.*" —COLONIAL CHURCH CHRONICLE.

THE WORSHIP OF GOD

AND FELLOWSHIP AMONG MEN.

A Series of Sermons on Public Worship. Fcap. 8vo. 3*s.* 6*d.*

I. Preaching, a Call to Worship. By Rev. F. D. MAURICE.—II. Common Prayer, the Method of Worship. By Rev. T. J. ROWSELL.—III. Baptism, an admission to the Privilege of Worship. By Rev. J. LL. DAVIES.—IV. The Lord's Supper, the most Sacred Bond of Worship. By Rev. D. J. VAUGHAN.—V. The Sabbath Day, the Refreshment of Worship. By Rev. J. LL. DAVIES.—VI. The Bible, the Revelation of the Beginning and End of Worship. By Rev. F. D. MAURICE.

BY THOMAS RAWSON BIRKS, M.A.,

Rector of Kelshall, Examining Chaplain to the Lord Bishop of Carlisle; Author of "The Life of the Rev. E. Bickersteth."

The Difficulties of Belief, in connexion with the Creation and the Fall. Crown 8vo. cloth, 4*s.* 6*d.*

"*A profound and masterly essay.*"—ECLECTIC.

"*His arguments are original, and carefully and logically elaborated. We may add that they are distinguished by a marked sobriety and reverence for the Word of God.*"—RECORD.

PUBLISHED

SERMONS PREAC]

By the Rev. EDWARD

" We desire very highly t
boy's life and trials in a
and impressiveness. Th
kind."—LITERARY CHUR

BY DA

Professor of English

Essays, Biographical
Poets.

1. Shakespeare and Goeth
Devils: Luther's, Milton's,
ture of the Restoration.—
the Year 1770.—VII. Wor
Literature.—IX. Theories

" *Distinguished by a remark*
facts on which speculati
These Essays should be po

BY IS

Author of " Th

The Restora

"*A volume which contains lo*
as the magnanimity and
work bearing on religious
'The Restoration of Bel
books of the nineteenth
to the 'Analogy' of Butle
of Pascal in that of the

BY JOHN H

The Construction of
the Practical Applicati
Iron Girders. With

" *The great defect of all wo*
theoretical: the great m
Practice than Theory...
investigations into Girde
of the most valuable c
engineering."—ATHENÆ

THE WORKS OF

FREDERICK DENISON MAURICE, M.A.,

Chaplain of Lincoln's Inn.

Exposition of the Holy Scriptures:

(1.) The Patriarchs and Lawgivers. 6*s.*
(2.) The Prophets and Kings. 10*s.* 6*d.*
(3.) The Gospels of St. Matthew, Mark, and Luke, and the Epistles of St. Paul, Peter, James, and Jude. 14*s.*
(4.) The Gospel of St. John. 10*s.* 6*d.*
(5.) The Epistles of St. John. 7*s.* 6*d.*

Exposition of the Prayer Book:

(1.) Sermons on the Ordinary Services. 5*s.* 6*d.*
(2.) The Church a Family: Being Sermons on the Occasional Services. 4*s.* 6*d.*

Ecclesiastical History. 10*s.* 6*d.*

The Lord's Prayer. Third Edition. 2*s.* 6*d.*

The Doctrine of Sacrifice. 7*s.* 6*d.*

Theological Essays. Second Edition. 10*s.* 6*d.*

Christmas Day, and other Sermons. 10*s.* 6*d.*

The Religions of the World. Third Edition. 5*s.*

Learning and Working. 5*s.*

The Indian Crisis. Five Sermons. 2*s.* 6*d.*

The Sabbath, and other Sermons. Fcp. 8vo. cloth, 2*s.* 6*d.*

Law on the Fable of the Bees. Fcp. 8vo. cloth, 4*s.* 6*d.*

The Worship of the Church. A Witness for the Redemption of the World. 1*s.*

The Word "Eternal" and the Punishment of the Wicked. Third Edition. 1*s.*

Eternal Life and Eternal Death. 1*s.* 6*d.*

The Name Protestant, and the English Bishopric at Jerusalem. Second Edition. 3*s.*

Right and Wrong Methods of Supporting Protestantism. 1*s.*

The Duty of a Protestant in the Oxford Election. 1847. 1*s.*

The Case of Queen's College, London. 1*s.* 6*d.*

Death and Life. In Memoriam C.B.M. 1*s.*

Administrative Reform. 3*d.*

PUBLISHED BY M

MANUALS FOR THE

UNIFORMLY PRI

It is now about seven years s was first issued. Four others are in an advance volumes have met with, ha which the Publishers comm in the belief, that their ai prehensive, and accurate,' Student and interesting t unsuccessfully fulfilled.

The following paragraphs appeared be here conveniently reprod

"The Authors being Clergymen being designed primarily fc her Ministry, the books wi spirit and principles; and English Church teach cha opinions and principles of be made to avoid acrimony

"It will be the aim of the writer dogmatic expression of dout

A General View of the I New Testament durin

By Brooke Foss Westco College, Cambridge.

The Author is one of those who storehouses of German theology, sphere: and to recognise the va track the vagaries of their theoret the clear sight and sound feelin the best and most complete book well placed on the lists of our exa

"*Learned, dispassionate, discrimin state of Christian Literature in r*

"*To the student in Theology it w others who have any curiosity on most useful and instructive piec supply.*"—London Quarterly

THEOLOGICAL MANUALS—continued.

II.

History of the Christian Church, during the Middle Ages and the Reformation (A.D. 590–1600).

By Archdeacon Hardwick.

2 vols. crown 8vo. 10*s.* 6*d.* each.

Vol. I. History of the Church to the Excommunication of Luther. With Four Maps.

Vol. II. History of the Reformation.

Each Volume may be had separately.

"*Full in references and authority, systematic and formal in division, with enough of life in the style to counteract the dryness inseparable from its brevity, and exhibiting the results rather than the principles of investigation.* Mr. Hardwick *is to be congratulated on the successful achievement of a difficult task.*" —Christian Remembrancer.

"*He has bestowed patient and extensive reading on the collection of his materials; he has selected them with judgment; and he presents them in an equable and compact style.*"—Spectator.

"*To a good method and good materials* Mr. Hardwick *adds that great virtue, a perfectly transparent style. We did not expect to find great literary qualities in such a manual, but we* have *found them; we should be satisfied in this respect with conciseness and intelligibility; but while this book has both, it is also elegant, highly finished, and highly interesting.*"—Nonconformist.

III.

History of the Book of Common Prayer,

together with a Rationale of the several Offices. By Francis Procter, M.A., Vicar of Witton, Norfolk, formerly Fellow of St. Catharine's College, Cambridge. Third Edition, revised and enlarged. Crown 8vo. cloth, 10*s.* 6*d.*

"Mr. Procter's 'History of the Book of Common Prayer' *is by far the best commentary extant . . . . . . Not only do the present illustrations embrace the whole range of original sources indicated by* Mr. Palmer, *but* Mr. Procter *compares the present Book of Common Prayer with the Scotch and American forms; and he frequently sets out in full the Sarum Offices. As a manual of extensive information, historical and ritual, imbued with sound Church principles, we are entirely satisfied with* Mr. Procter's *important volume.*"

Christian Remembrancer.

"*It is indeed a complete and fairly-written history of the Liturgy; and from the dispassionate way in which disputed points are touched on, will prove to many troubled consciences what ought to be known to them, viz.:—that they may, without fear of compromising the principles of evangelical truth, give their assent and consent to the contents of the Book of Common Prayer.* Mr. Procter *has done a great service to the Church by this admirable digest.*"

Church of England Quarterly

CLASS-BOOKS FOR COLLEGES AND SCHOOLS,

PUBLISHED BY

MACMILLAN AND CO.

CAMBRIDGE,

AND 23, HENRIETTA STREET, COVENT GARDEN, LONDON.

MATHEMATICAL.

BY G. B. AIRY, M.A., F.R.S.,

Astronomer Royal.

Mathematical Tracts on the Lunar and Planetary Theories. The Figure of the Earth, Precession and Nutation, the Calculus of Variations, and the Undulatory Theory of Optics. Fourth Edition, revised and improved. 8vo. cloth, 15*s*.

BY R. D. BEASLEY, M.A.

Head Master of Grantham Grammar School.

An Elementary Treatise on Plane Trigonometry; with a numerous Collection of Examples, chiefly designed for the use of Schools and Beginners. Crown 8vo. cloth, 3*s*. 6*d*.

BY GEORGE BOOLE, LL.D.,

Professor of Natural Philosophy in Queen's College, Cork.

A Treatise on Differential Equations. Crown 8vo. cloth, 14*s*.

BY W. H. DREW, M.A.,

Second Master of Blackheath Proprietary School.

A Geometrical Treatise on Conic Sections. With a Copious Collection of Examples, embodying every Question which has been proposed in the Senate-House at Cambridge. Crown 8vo. cloth, 4*s*. 6*d*.

BY HUGH GODFRAY, M.A.,

St. John's College, Cambridge.

An Elementary Treatise on the Lunar Theory. With a brief Sketch of the History of the Problem up to the time of Newton. 8vo. cloth, 5*s*. 6*d*.

BY A. R. GRANT, M.A.,

H. M. Inspector of Schools.

Plane Astronomy. Including Explanations of Celestial Phenomena, and Descriptions of Astronomical Instruments. 8vo. 6s.

PUBLISHED BY M

BY J. C. S

Fellow of St. Jo

1. The Elements of Plane
Greatly improved and enla
Ninth Edition, Crow

2. An Introduction to th
metry. Designed for t

3. The Cambridge Cour
and Hydrostatics. T
and Problems chiefly from
with Hints for their Soluti

By P. G. TAIT, M.A., Fellow
of Mathematics in Queen's
Fellow of St. Peter's Colleg

A Treatise on Dynamics

BY S. PAR

Fellow and Assistant Tutor

An Elementary Treatise
the Junior Classes at the
Schools. With a copious C

BY J. B.

Fellow of Clare

Elementary Hydrostatics.
Solutions. Second Edition.

BY I. TODHUNTER, M.A.,

Fellow and Assistant Tutor of St. John's College, Cambridge.

1. A Treatise on the Differential Calculus. With numerous Examples. Second Edition. Crown 8vo. cloth, 10*s.* 6*d.*

2. A Treatise on the Integral Calculus, and its Applications. With numerous Examples.
Crown 8vo. cloth, 10*s.* 6*d.*

3. A Treatise on Analytical Statics. With numerous Examples. Second Edition.
Crown 8vo. cloth, 10*s.* 6*d.*

4. A Treatise on Plane Co-ordinate Geometry, as applied to the Straight Line and the CONIC SECTIONS. With numerous Examples. Second Edition.
Crown 8vo. cloth, 10*s.* 6*d.*

5. A Treatise on Algebra. For the use of Students in the Universities and in Schools. Crown 8vo. cloth, 7*s.* 6*d.*

6. Examples of Analytical Geometry of Three Dimensions. Crown 8vo. cloth, 4*s.*

7. Plane Trigonometry. For Schools and Colleges.
Crown 8vo. cloth. 7*s.* 6*d.*

BY W. P. WILSON, M.A.,

Professor of Mathematics in the University of Melbourne.

A Treatise on Dynamics. 8vo. bds. 9*s.* 6*d.*

BY J. WRIGHT, M.A.,

Of Trinity College, Cambridge, Head Master of Sutton Coldfield Grammar School.

1. A Help to Latin Grammar. With Easy Exercises, both English and Latin, Questions, and Vocabulary.

Crown 8vo. cloth, 4*s*. 6*d*.

"*This book aims at helping the learner to overstep the threshold difficulties of the Latin Grammar; and never was there a better aid offered alike to teacher and scholar in that arduous pass. The style is at once familiar and strikingly simple and lucid; and the explanations precisely hit the difficulties, and thoroughly explain them. It is exactly adapted for the instruction of children; and will, we prophecy, be the means of making many a good Latin scholar. The children who are early disgusted by heaps of rules which they cannot understand is legion. It is a great detriment to good instruction, and Mr. Wright deserves our best thanks or removing it. No child of moderate capacity can fail to understand* his *grammar, the study of which ought to precede that of every other. It will also much facilitate the acquirement of English Grammar.*"—English Journal of Education.

2. The Seven Kings of Rome. An easy Narrative, abridged from the First Book of Livy, by the omission of difficult passages, in order to serve as a First Latin Construing-book, with Grammatical Notes and Index. Second Edition. Fcap. 8vo. cloth, 3*s*.

"*The Notes are abundant, explicit, and full of such grammatical and other information as boys require.*"—Athenæum.

3. A Vocabulary and Exercises on "The Seven Kings of Rome." Fcap. 8vo. cloth, 2*s*. 6*d*.

*** *The Vocabulary may be obtained bound up with* "The Seven Kings of Rome," *price* 5*s*.

4. Hellenica; or, a History of Greece in Greek, beginning with the Invasion of Xerxes; as related by Diodorus and Thucydides. With Notes, Critical and Historical, and a Vocabulary, in order to serve as a First Greek Construing-book. Second Edition. 12mo. cloth, 3*s*. 6*d*.

"*The Notes are exactly of that illustrative and suggestive nature which the student at the commencement of his course most stands in need of, and which the scholar, who is also an experienced teacher, alone can supply.*"—Educational Times.

A good plan well executed."—Guardian.

RELIGIOUS.

History of the Christian Church, from Gregory the Great to the Reformation (A.D. 590–1600). By CHARLES HARDWICK, M.A., Christian Advocate in the University of Cambridge. Two Vols. crown 8vo. cloth, 21*s.*

Vol. I. History from Gregory the Great to the Excommunication of Luther. With Maps.

Vol. II. History of the Reformation in the Church.

Each Volume may be had separately, price 10*s.* 6*d.*

History of the Book of Common Prayer: with a Rationale of its Offices. By FRANCIS PROCTER, M.A., Vicar of Witton, Norfolk, and late Fellow of St. Catherine's College. Third Edition, revised and enlarged. Crown 8vo. cloth, 10*s.* 6*d.*

Notes for Lectures on Confirmation. With suitable Prayers. By C. J. VAUGHAN, D.D., Head Master of Harrow School. Fcap. 8vo. cloth, 1*s.* 6*d.*

The Catechiser's Manual; or, The Church Catechism Illustrated and Explained. By ARTHUR RAMSAY, M.A., of Trinity College, Cambridge. 18mo. cloth, 3*s.* 6*d.*

Hand-Book to Butler's Analogy. With Notes. By C. A. SWAINSON, M.A., Principal of the Theological College and Prebendary of Chichester. Crown 8vo. 1*s.* 6*d.*

History of the Canon of the New Testament during the First Four Centuries. By BROOKE FOSS WESTCOTT, M.A., Assistant Master of Harrow School; late Fellow of Trinity College, Cambridge. Crown 8vo. cloth, 12*s.* 6*d.*

History of the Christian Church during the First Three Centuries, and the Reformation in England. By WILLIAM SIMPSON, M.A., of Queen's College, Cambridge. Fcp. 8vo. cloth, 5*s.*

Analysis of Paley's Evidences of Christianity, in the form of Question and Answer, with Examination Papers. By CHARLES H. CROSSE, M.A., of Caius College, Cambridge. 18mo. 3*s.* 6*d.*

R. CLAY, PRINTER, BREAD STREET HILL.

[redacted]

[redacted]